



Lesson 9: Radicals and Conjugates

Student Outcomes

- Students understand that the sum of two square roots (or two cube roots) is not equal to the square root (or cube root) of their sum.
- Students convert expressions to simplest radical form.
- Students understand that the product of conjugate radicals can be viewed as the difference of two squares.

Lesson Notes

Because this lesson deals with radicals, it might seem out of place amid other lessons on polynomials. A major theme, however, is the parallelism between the product of conjugate radicals and the difference of two squares. There is also parallelism between taking the square root or cube root of a sum and taking the square of a sum; both give rise to an error sometimes called *the freshman's dream* or *the illusion of linearity*. If students are not careful, they may easily conclude that $(x + y)^n$ and $x^n + y^n$ are equivalent expressions for all $n \geq 0$, when this is only true for $n = 1$. Additionally, this work with radicals prepares students for later work with radical expressions in Module 3. Throughout this lesson, students employ MP.7, as they see complicated expressions as being composed of simpler ones. Additionally, the Opening Exercise offers further practice in making a conjecture (MP.3).

Scaffolding:

If necessary, remind students that $\sqrt{2}$ and $-\sqrt[3]{4}$ are irrational numbers. They cannot be written in the form $\frac{p}{q}$ for integers p and q . They are numbers, however, and, just like $\sqrt{4}$ and $-\sqrt[3]{64}$, can be found on a number line. On the number line, $\sqrt{2} = 1.414\dots$ is just to the right of 1.4, and $-\sqrt[3]{4} = -1.5874\dots$ is just to the left of -1.587 .

Classwork

Opening Exercise (3 minutes)

The Opening Exercise is designed to show students that they need to be cautious when working with radicals. The multiplication and division operations combine with radicals in a predictable way, but the addition and subtraction operations do not.

The square root of a product is the product of the two square roots. For example,

$$\begin{aligned}\sqrt{4 \cdot 9} &= \sqrt{2 \cdot 2 \cdot 3 \cdot 3} \\ &= \sqrt{6 \cdot 6} \\ &= 6 \\ &= 2 \cdot 3 \\ &= \sqrt{4} \cdot \sqrt{9}.\end{aligned}$$

Similarly, the square root of a quotient is the quotient of the two square roots: $\sqrt{\frac{25}{16}} = \frac{5}{4} = \frac{\sqrt{25}}{\sqrt{16}}$. And the same holds true for multiplication and division with cube roots, but not for addition or subtraction with square or cube roots.

Begin by posing the following question for students to work on in groups.

Opening Exercise

Which of these statements are true for all $a, b > 0$? Explain your conjecture.

i. $2(a + b) = 2a + 2b$

ii. $\frac{a + b}{2} = \frac{a}{2} + \frac{b}{2}$

iii. $\sqrt{a + b} = \sqrt{a} + \sqrt{b}$

Scaffolding:

If necessary, circulate to help students get started by suggesting that they substitute numerical values for a and b . Perfect squares like 1 and 4 are good values to start with when square roots are in the expression.

Discussion (3 minutes)

Students should be able to show that the first two equations are true for all a and b , and they should be able to find values of a and b for which the third equation is not true. (In fact, it is always false, as students will show in the Problem Set.)

- Can you provide cases for which the third equation is not true? (Remind them that a single counterexample is sufficient to make an equation untrue in general.)
 - *Students should give examples such as $\sqrt{9} + \sqrt{16} = 3 + 4 = 7$, but $\sqrt{9 + 16} = \sqrt{25} = 5$.*

Point out that just as they have learned (in Lesson 2, if not before) that the square of $(x + y)$ is not equal to the sum of x^2 and y^2 (for $x, y \neq 0$), so it is also true that the square root of $(x + y)$ is not equal to the sum of the square roots of x and y (for $x, y > 0$). Similarly, the cube root of $(x + y)$ is not equal to the sum of the cube roots of x and y (for $x, y \neq 0$).

Example 1 (2 minutes)

Explain to students that an expression is in *simplest radical form* when the radicand (the expression under the radical sign) has no factor that can be raised to a power greater than or equal to the index (either 2 or 3), and there is no radical in the denominator. Present the following example.

Example 1

Express $\sqrt{50} - \sqrt{18} + \sqrt{8}$ in simplest radical form and combine like terms.

$$\sqrt{50} = \sqrt{25 \cdot 2} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$$

$$\sqrt{18} = \sqrt{9 \cdot 2} = \sqrt{9} \cdot \sqrt{2} = 3\sqrt{2}$$

$$\sqrt{8} = \sqrt{4 \cdot 2} = \sqrt{4} \cdot \sqrt{2} = 2\sqrt{2}$$

Therefore, $\sqrt{50} - \sqrt{18} + \sqrt{8} = 5\sqrt{2} - 3\sqrt{2} + 2\sqrt{2} = 4\sqrt{2}$.

Scaffolding:

If necessary, ask students about expressions that are easier to express in simplest radical form, such as:

$$\sqrt{8}$$

$$\sqrt{6}$$

$$\sqrt{18}$$

$$\sqrt{20}$$

$$\sqrt{20} + \sqrt{8}$$

$$\sqrt{20} - \sqrt{18}$$

Exercises 1–5 (8 minutes)

The following exercises make use of the rules for multiplying and dividing radicals. Express each expression in simplest radical form and combine like terms.

Scaffolding:

Circulate to make sure that students understand that they are looking for perfect square factors when the index is 2 and perfect cube factors when the index is 3. Remind students that the cubes of the first counting numbers are 1, 8, 27, 64, and 125.

Exercises 1–5

<p>1. $\sqrt{\frac{1}{4}} + \sqrt{\frac{9}{4}} - \sqrt{45}$ $2 - 3\sqrt{5}$</p>	<p>2. $\sqrt{2}(\sqrt{3} - \sqrt{2})$ $\sqrt{6} - 2$</p>
<p>3. $\sqrt{\frac{3}{8}}$ $\frac{\sqrt{3}}{\sqrt{8}} = \frac{\sqrt{6}}{\sqrt{16}} = \frac{\sqrt{6}}{4}$</p>	<p>4. $\sqrt[3]{\frac{5}{32}}$ $\frac{\sqrt[3]{5}}{\sqrt[3]{32}} = \frac{\sqrt[3]{10}}{\sqrt[3]{64}} = \frac{\sqrt[3]{10}}{4}$</p>
<p>5. $\sqrt[3]{16x^5}$ $\sqrt[3]{8x^3} \cdot \sqrt[3]{2x^2} = 2x\sqrt[3]{2x^2}$</p>	

In the example and exercises above, we repeatedly used the following properties of radicals (write the following statements on the board).

$$\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$$

$$\sqrt[3]{a} \cdot \sqrt[3]{b} = \sqrt[3]{ab}$$

$$\frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}}$$

$$\frac{\sqrt[3]{a}}{\sqrt[3]{b}} = \sqrt[3]{\frac{a}{b}}$$

- When do these identities make sense?
 - *Students should answer that the identities make sense for the square roots whenever $a \geq 0$ and $b \geq 0$, with $b \neq 0$ when b is a denominator. They make sense for the cube roots for all a and b , with $b \neq 0$ when b is a denominator.*

Example 2 (8 minutes)

This example is designed to introduce conjugates and their properties.

Example 2

Multiply and combine like terms. Then explain what you notice about the two different results.

$(\sqrt{3} + \sqrt{2})(\sqrt{3} + \sqrt{2})$

$(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})$

Solution (with teacher comments and a question):

The first product is $\sqrt{3} \cdot \sqrt{3} + 2(\sqrt{3} \cdot \sqrt{2}) + \sqrt{2} \cdot \sqrt{2} = 5 + 2\sqrt{6}$.

The second product is $\sqrt{3} \cdot \sqrt{3} - (\sqrt{3} \cdot \sqrt{2}) + (\sqrt{3} \cdot \sqrt{2}) - \sqrt{2} \cdot \sqrt{2} = 3 - 2 = 1$.

The first product is an irrational number; the second is an integer.

The second product has the nice feature that the radicals have been eliminated. In that case, the two factors are given a special name: two binomials of the form $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are called *conjugate radicals*:

$\sqrt{a} + \sqrt{b}$ is the conjugate of $\sqrt{a} - \sqrt{b}$, and

$\sqrt{a} - \sqrt{b}$ is the conjugate of $\sqrt{a} + \sqrt{b}$.

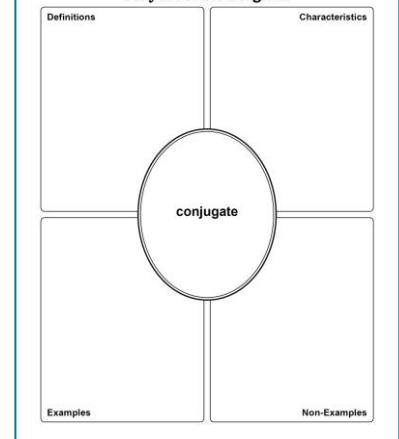
More generally, for any expression in two terms, at least one of which contains a radical, its conjugate is an expression consisting of the same two terms but with the opposite sign separating the terms. For example, the conjugate of $2 - \sqrt{3}$ is $2 + \sqrt{3}$, and the conjugate of $\sqrt[3]{5} + \sqrt{3}$ is $\sqrt[3]{5} - \sqrt{3}$.

- What polynomial identity is suggested by the product of two conjugates?
 - *Students should answer that it looks like the difference of two squares. The product of two conjugates has the form of the difference of squares: $(x + y)(x - y) = x^2 - y^2$.*

The following exercise focuses on the use of conjugates.

Scaffolding:

Students may have trouble with the word *conjugate*. If so, have them fill out the following diagram.



Exercise 6 (5 minutes)

Exercise 6

6. Find the product of the conjugate radicals.

$(\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3})$	$5 - 3 = 2$
$(7 + \sqrt{2})(7 - \sqrt{2})$	$49 - 2 = 47$
$(\sqrt{5} + 2)(\sqrt{5} - 2)$	$5 - 4 = 1$

- In each case in Exercise 6, is the result the difference of two squares?
 - *Yes. For example, if we think of 5 as $(\sqrt{5})^2$ and 3 as $(\sqrt{3})^2$, then $5 - 3 = (\sqrt{5})^2 - (\sqrt{3})^2$.*

Example 3 (6 minutes)

This example is designed to show how division by a radical can be reduced to division by an integer by multiplication by the conjugate radical in the numerator and denominator.

Example 3

Write $\frac{\sqrt{3}}{5-2\sqrt{3}}$ in simplest radical form.

$$\frac{\sqrt{3}}{5-2\sqrt{3}} = \frac{\sqrt{3}}{5-2\sqrt{3}} \cdot \frac{5+2\sqrt{3}}{5+2\sqrt{3}} = \frac{\sqrt{3}(5+2\sqrt{3})}{25-12} = \frac{5\sqrt{3}+6}{13}$$

The process for simplifying an expression with a radical in the denominator has two steps:

1. Multiply the numerator and denominator of the fraction by the *conjugate* of the denominator.
2. Simplify the resulting expression.

Closing (5 minutes)

- Radical expressions with the same index and same radicand combine in the same way as like terms in a polynomial when performing addition and subtraction.

For example, $\sqrt[3]{3} + \sqrt{2} + 5\sqrt[3]{3} - \sqrt{7} + \sqrt{3} + \sqrt{7} + 3\sqrt{2} = 6\sqrt[3]{3} + 4\sqrt{2} + \sqrt{3}$.

- Simplifying an expression with a radical in the denominator relies on an application of the difference of squares formula.

- For example, to simplify $\frac{3}{\sqrt{2}+\sqrt{3}}$ we treat the denominator like a binomial.

Substitute $\sqrt{2} = x$ and $\sqrt{3} = y$, and then

$$\frac{3}{\sqrt{2}+\sqrt{3}} = \frac{3}{x+y} = \frac{3}{x+y} \cdot \frac{x-y}{x-y} = \frac{3(x-y)}{x^2-y^2}$$

Since $x = \sqrt{2}$ and $y = \sqrt{3}$, $x^2 - y^2$ is an integer. In this case, $x^2 - y^2 = -1$.

$$\frac{3}{\sqrt{2}+\sqrt{3}} \cdot \frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}} = \frac{3(\sqrt{2}-\sqrt{3})}{2-3} = -3(\sqrt{2}-\sqrt{3})$$

Ask students to summarize the important parts of the lesson, either in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson. The following are some important summary elements.

Lesson Summary

- For real numbers $a \geq 0$ and $b \geq 0$, where $b \neq 0$ when b is a denominator,

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \text{ and } \sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

- For real numbers $a \geq 0$ and $b \geq 0$, where $b \neq 0$ when b is a denominator,

$$\sqrt[3]{ab} = \sqrt[3]{a} \cdot \sqrt[3]{b} \text{ and } \sqrt[3]{\frac{a}{b}} = \frac{\sqrt[3]{a}}{\sqrt[3]{b}}$$

- Two binomials of the form $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are called conjugate radicals:

$\sqrt{a} + \sqrt{b}$ is the conjugate of $\sqrt{a} - \sqrt{b}$, and

$\sqrt{a} - \sqrt{b}$ is the conjugate of $\sqrt{a} + \sqrt{b}$.

For example, the conjugate of $2 - \sqrt{3}$ is $2 + \sqrt{3}$.

- To rewrite an expression with a denominator of the form $\sqrt{a} + \sqrt{b}$ in simplest radical form, multiply the numerator and denominator by the conjugate $\sqrt{a} - \sqrt{b}$ and combine like terms.

Exit Ticket (5 minutes)

Name _____

Date _____

Lesson 9: Radicals and Conjugates

Exit Ticket

- Rewrite each of the following radicals as a rational number or in simplest radical form.
 - $\sqrt{49}$
 - $\sqrt[3]{40}$
 - $\sqrt{242}$
- Find the conjugate of each of the following radical expressions.
 - $\sqrt{5} + \sqrt{11}$
 - $9 - \sqrt{11}$
 - $\sqrt[3]{3} + 1.5$
- Rewrite each of the following expressions as a rational number or in simplest radical form.
 - $\sqrt{3}(\sqrt{3} - 1)$
 - $(5 + \sqrt{3})^2$
 - $(10 + \sqrt{11})(10 - \sqrt{11})$

Exit Ticket Sample Solution

1. Rewrite each of the following radicals as a rational number or in simplest radical form.	
a. $\sqrt{49}$	7
b. $\sqrt[3]{40}$	$2\sqrt[3]{5}$
c. $\sqrt{242}$	$11\sqrt{2}$
2. Find the conjugate of each of the following radical expressions.	
a. $\sqrt{5} + \sqrt{11}$	$\sqrt{5} - \sqrt{11}$
b. $9 - \sqrt{11}$	$9 + \sqrt{11}$
c. $\sqrt[3]{3} + 1.5$	$\sqrt[3]{3} - 1.5$
3. Rewrite each of the following expressions as a rational number or in simplest radical form.	
a. $\sqrt{3}(\sqrt{3} - 1)$	$3 - \sqrt{3}$
b. $(5 + \sqrt{3})^2$	$28 + 10\sqrt{3}$
c. $(10 + \sqrt{11})(10 - \sqrt{11})$	89

Problem Set Sample Solutions

Problem 10 is different from the others and may require some discussion and explanation before students work on it. Consider explaining that the converse of an *if-then* theorem is obtained by interchanging the clauses introduced by *if* and *then*, and that the converse of such a theorem is not necessarily a valid theorem. The converse of the Pythagorean theorem will be important for the development of a formula leading to Pythagorean triples in Lesson 10.

1. Express each of the following as a rational number or in simplest radical form. Assume that the symbols a , b , and x represent positive numbers.	
a. $\sqrt{36}$	6
b. $\sqrt{72}$	$6\sqrt{2}$
c. $\sqrt{18}$	$3\sqrt{2}$
d. $\sqrt{9x^3}$	$3x\sqrt{x}$

e.	$\sqrt{27x^2}$	$3x\sqrt{3}$
f.	$\sqrt[3]{16}$	$2\sqrt[3]{2}$
g.	$\sqrt[3]{24a}$	$2\sqrt[3]{3a}$
h.	$\sqrt{9a^2 + 9b^2}$	$3\sqrt{a^2 + b^2}$
2. Express each of the following in simplest radical form, combining terms where possible.		
a.	$\sqrt{25} + \sqrt{45} - \sqrt{20}$	$5 + \sqrt{5}$
b.	$3\sqrt{3} - \sqrt{\frac{3}{4}} + \sqrt{\frac{1}{3}}$	$\frac{17\sqrt{3}}{6}$
c.	$\sqrt[3]{54} - \sqrt[3]{8} + 7\sqrt[3]{\frac{1}{4}}$	$\frac{13\sqrt[3]{2}}{2} - 2$
d.	$\sqrt[3]{\frac{5}{8}} + \sqrt[3]{40} - \sqrt[3]{\frac{8}{9}}$	$\frac{5\sqrt[3]{5}}{2} - \frac{2\sqrt[3]{3}}{3}$
3. Evaluate $\sqrt{x^2 - y^2}$ when $x = 33$ and $y = 15$.		
		$12\sqrt{6}$
4. Evaluate $\sqrt{x^2 + y^2}$ when $x = 20$ and $y = 10$.		
		$10\sqrt{5}$
5. Express each of the following as a rational expression or in simplest radical form. Assume that the symbols x and y represent positive numbers.		
a.	$\sqrt{3}(\sqrt{7} - \sqrt{3})$	$\sqrt{21} - 3$
b.	$(3 + \sqrt{2})^2$	$11 + 6\sqrt{2}$
c.	$(2 + \sqrt{3})(2 - \sqrt{3})$	1
d.	$(2 + 2\sqrt{5})(2 - 2\sqrt{5})$	-16
e.	$(\sqrt{7} - 3)(\sqrt{7} + 3)$	-2
f.	$(3\sqrt{2} + \sqrt{7})(3\sqrt{2} - \sqrt{7})$	11
g.	$(x - \sqrt{3})(x + \sqrt{3})$	$x^2 - 3$
h.	$(2x\sqrt{2} + y)(2x\sqrt{2} - y)$	$8x^2 - y^2$

6. Simplify each of the following quotients as far as possible.

a. $(\sqrt{21} - \sqrt{3}) \div \sqrt{3}$ $\sqrt{7} - 1$

b. $(\sqrt{5} + 4) \div (\sqrt{5} + 1)$ $\frac{1}{4}(1 + 3\sqrt{5})$

c. $(3 - \sqrt{2}) \div (3\sqrt{2} - 5)$ $-\frac{1}{7}(9 + 4\sqrt{2})$

d. $(2\sqrt{5} - \sqrt{3}) \div (3\sqrt{5} - 4\sqrt{2})$ $\frac{1}{13}(30 - 3\sqrt{15} + 8\sqrt{10} - 4\sqrt{6})$

7. If $x = 2 + \sqrt{3}$, show that $x + \frac{1}{x}$ has a rational value.

$$x + \frac{1}{x} = 4$$

8. Evaluate $5x^2 - 10x$ when the value of x is $\frac{2-\sqrt{5}}{2}$.

$$\frac{5}{4}$$

9. Write the factors of $a^4 - b^4$. Express $(\sqrt{3} + \sqrt{2})^4 - (\sqrt{3} - \sqrt{2})^4$ in a simpler form.

Factors: $(a^2 + b^2)(a + b)(a - b)$

Simplified form: $40\sqrt{6}$

10. The converse of the Pythagorean theorem is also a theorem: If the square of one side of a triangle is equal to the sum of the squares of the other two sides, then the triangle is a right triangle.

Use the converse of the Pythagorean theorem to show that for $A, B, C > 0$, if $A + B = C$, then $\sqrt{A} + \sqrt{B} > \sqrt{C}$, so that $\sqrt{A} + \sqrt{B} > \sqrt{A + B}$.

Solution 1: Since $A, B, C > 0$, we can interpret these quantities as the areas of three squares whose sides have lengths \sqrt{A} , \sqrt{B} , and \sqrt{C} . Because $A + B = C$, then by the converse of the Pythagorean theorem, \sqrt{A} , \sqrt{B} , and \sqrt{C} are the lengths of the legs and hypotenuse of a right triangle. In a triangle, the sum of any two sides is greater than the third side. Therefore, $\sqrt{A} + \sqrt{B} > \sqrt{C}$, so $\sqrt{A} + \sqrt{B} > \sqrt{A + B}$.

Solution 2: Since $A, B, C > 0$, we can interpret these quantities as the areas of three squares whose sides have lengths \sqrt{A} , \sqrt{B} , and \sqrt{C} . Because $A + B = C$, then by the converse of the Pythagorean theorem, $\sqrt{A} = a$, $\sqrt{B} = b$, and $c = \sqrt{C}$ are the lengths of the legs and hypotenuse of a right triangle, so $a, b, c > 0$. Therefore, $2ab > 0$. Adding equal positive quantities to each side of that inequality, we get $a^2 + b^2 + 2ab > c^2$, which we can rewrite as $(a + b)^2 > c^2$. Taking the positive square root of each side, we get $a + b > c$, or equivalently, $\sqrt{A} + \sqrt{B} > \sqrt{C}$. We then have $\sqrt{A} + \sqrt{B} > \sqrt{A + B}$.