Lesson 21: Modeling Riverbeds with Polynomials

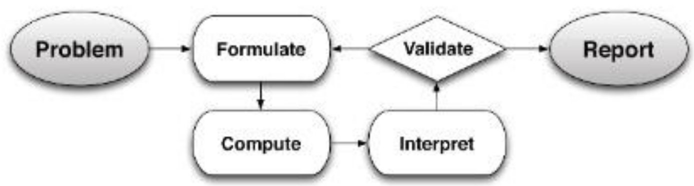
Student Outcomes

- Students model a cross-section of a riverbed with a polynomial function and estimate fluid flow with their algebraic model.

Lesson Notes

This is the second half of a two-day modeling lesson. In previous modeling lessons, students relied on the graphing calculator to find polynomial functions that fit a set of data, and in the previous lesson the polynomial was found algebraically. In this lesson, we use the website Wolfram Alpha (www.wolframalpha.com) to find the polynomial to fit the data. If students do not have access to computers or the Internet during class, the polynomial can be found using quartic regression on a graphing calculator.

The previous lesson introduced the problem of modeling the shape of a riverbed and computing the volumetric flow. In this lesson, students can actually do the modeling, with the help of technology, and then interpret the results and create a report detailing their findings, thus completing the modeling cycle.



Classwork

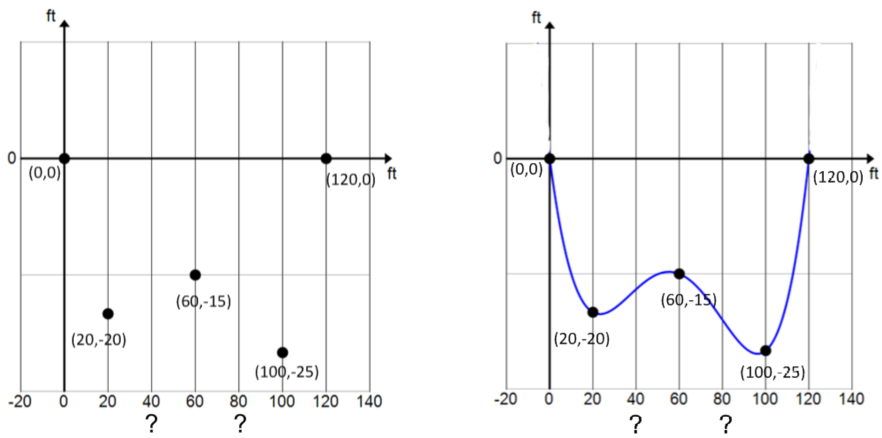
Opening (3 minutes)

Review the Mathematical Modeling Exercise with students. In this lesson, students:

1. Find a function that models the shape of the riverbed based on the five data points given in the graph below.
2. Approximate the area of the cross-sectional region using triangles and trapezoids.
3. Calculate the volumetric flow rate of the water in gallons per minute.
4. Create a report of the findings.

Mathematical Modeling Exercise

The Environmental Protection Agency (EPA) is studying the flow of a river in order to establish flood zones. The EPA hired a surveying company to determine the flow rate of the river, measured as volume of water per minute. The firm set up a coordinate system and found the depths of the river at five locations as shown on the graph below. After studying the data, the firm decided to model the riverbed with a polynomial function and divide the area into six regions that are either trapezoidal or triangular so that the overall area can be easily estimated. The firm needs to approximate the depth of the river at two more data points in order to do this.



Mathematical Modeling Exercises 1–5 (32 minutes)

Now return to the Mathematical Modeling Exercise.

- How many data points are we given? What is the lowest degree of a polynomial that passes through these points?
  - *We were given five data points, so we can model the data using a fourth-degree polynomial function.*
- We are going to use Wolfram Alpha to find the fourth-degree polynomial that fits the data. It follows a procedure based on the remainder theorem that we used in the previous two examples to find the equation.

MP.5

Scaffolding:

If students are overwhelmed with the new technology along with new content, most graphing calculators perform polynomial regression up to degree 4, so consider having students use that more familiar technological tool.

Go to www.wolframalpha.com. Type in the following command:

Interpolating polynomial $[\{(0,0),(20,-20),(60,-15),(100,-25),(120,0)\},x]$.

Students can find $P(40)$ and $P(80)$ either by substituting into the function displayed or by editing the command on Wolfram Alpha as follows:

Interpolating polynomial $[\{(0,0),(20,-20),(60,-15),(100,-25),(120,0)\},40]$

Interpolating polynomial $[\{(0,0),(20,-20),(60,-15),(100,-25),(120,0)\},80]$.

Allow students time to work through the remainder of the exercise either individually or in groups. The conversion from cubic feet per minute to gallons per minute can also be done using Wolfram Alpha.

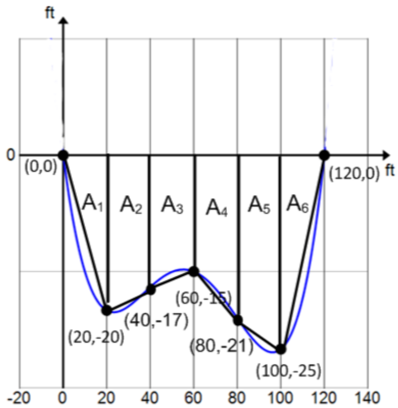
1. **Find a polynomial P that fits the five given data points.**

   $$P(x) = \frac{17}{3840000}x^4 - \frac{33}{32000}x^3 + \frac{751}{9600}x^2 - \frac{35}{16}x$$

2. **Use the polynomial to estimate the depth of the river at $x = 40$ and $x = 80$.**

   $P(40) = -17$ *and* $P(80) = -21$

3. **Estimate the area of the cross section.**



$$\begin{aligned} A &= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 \\ &= 200\text{ ft}^2 + 370\text{ ft}^2 + 320\text{ ft}^2 + 360\text{ ft}^2 + 460\text{ ft}^2 + 250\text{ ft}^2 \\ &= 1{,}960\text{ ft}^2 \end{aligned}$$

Suppose that the river flow speed was measured to be an average speed of $176\frac{\text{ft.}}{\text{min}}$ at the cross section.

4. **What is the volumetric flow of the water (the volume of water per minute)?**

   $$(1{,}960\text{ ft}^2)\left(176\frac{\text{ft.}}{\text{min}}\right) = 344{,}960\frac{\text{ft}^3}{\text{min}}$$

5. **Convert the flow to gallons per minute. [Note: 1 cubic foot ≈ 7.48052 gallons.]**

   $$2{,}580{,}480\ \frac{\text{gallons}}{\text{min}}$$

- How could the surveyors measure the speed of the water?
  - *They could time an object (like a ball) floating over a set distance.*
- What factors would need to be considered when measuring the flow?
  - *Water may flow faster below the surface. The flow rate may vary from the edge of the river to the middle of the river and around obstacles such as rocks.*
- Remember that the EPA is interested in identifying flood-prone areas. How might the information you have gathered help the EPA?
  - *If the normal volumetric flow of the river has been recorded by the EPA, then they can record the normal water levels in the river and use that knowledge to predict what level of volumetric flow would result in flooding. In cases of heavy rain, the EPA could identify if an area is likely to flood due to the increased volumetric flow.*

Closing (5 minutes)

Ask students to summarize the important parts of the lesson in writing, to a partner, or as a class. Use this as an opportunity to informally assess understanding of the lesson.

Exit Ticket (5 minutes)

Name ____ Date ____________________

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Exit Ticket

Explain the process you used to estimate the volumetric flow of the river, from accumulating the data to calculating the flow of water.

Exit Ticket Sample Solutions

Explain the process you used to estimate the volumetric flow of the river, from accumulating the data to calculating the flow of water.

We were given data points that represented the depth of the river at various points in the cross-section. We used Wolfram Alpha to find a polynomial P that passed through those given points. We used the polynomial P to find more data points. Then we approximated the area of the cross-section using triangles and trapezoids; figures whose area we know how to calculate. Once we had an approximate area of the cross-section, we multiplied that area by the speed of the water across the cross-section to find the amount of volumetric flow in units of cubic feet per minute. The last step was to convert cubic feet to gallons to get the amount of volumetric flow in units of gallons per minute.

Problem Set Sample Solutions

Problem 2 requires the use of a computer. This could be completed in class if students do not have access to computers.

1. **As the leader of the surveying team, write a short report to the EPA on your findings from the in-class exercises. Be sure to include data and calculations.**

   After collecting data at the site, we decided that the cross-sectional area could be approximated using trapezoids. In order to increase the accuracy of our area approximation, more data points were needed. We chose to use the data collected to model the riverbed using a degree 4 polynomial function. We used the computational knowledge engine Wolfram Alpha to find the polynomial that fit the data, which is $P(x) = \frac{17}{3\,840\,000}x^4 - \frac{33}{32\,000}x^3 + \frac{751}{9600}x^2 - \frac{35}{16}x$. Using the polynomial P, we were able to estimate enough data points to calculate the cross-sectional area and determined that it was $1,960\ \text{ft}^2$. Using this information and the average speed of the water at the cross-section, which was $\frac{\text{ft.}}{\text{min}}$, we were able to compute the volumetric flow of the river at that cross-section. We determined that the volumetric flow is approximately $344,960\ \frac{\text{ft}^3}{\text{min}}$, which is $2,580,480\ \frac{\text{gallons}}{\text{min}}$.

2. **Suppose that depths of the riverbed were measured for a different cross-section of the river.**

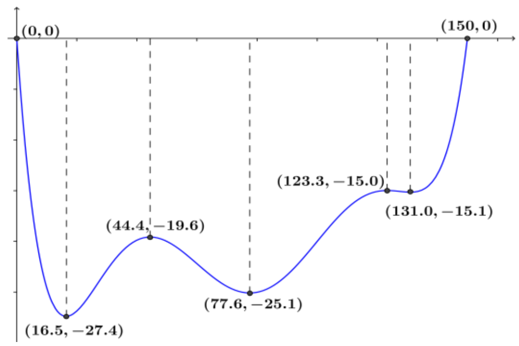
   a. **Use Wolfram Alpha to find the interpolating polynomial Q with values:**

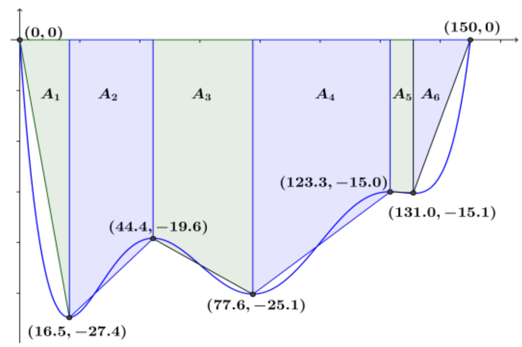
   $Q(0) = 0,$ $Q(16.5) = -27.4,$ $Q(44.4) = -19.6,$ $Q(77.6) = -25.1,$

   $Q(123.3) = -15.0,$ $Q(131.1) = -15.1,$ $Q(150) = 0.$

   $$Q(x) = \frac{7}{8\,789\,062\,500}x^6 - \frac{11}{29\,296\,875}x^5 + \frac{191}{2\,812\,500}x^4 - \frac{11}{1875}x^3 + \frac{2759}{11\,250}x^2 - \frac{329}{75}x$$

b. **Sketch the cross-section of the river, and estimate its area.**





Area estimates:

$$A_1 = \frac{1}{2}(27.4)(16.5) = 226.05$$

$$A_2 = \frac{1}{2}(27.4 + 19.6)(44.4 - 16.5) = 655.65$$

$$A_3 = \frac{1}{2}(25.1 + 19.6)(77.6 - 44.4) = 742.02$$

$$A_4 = \frac{1}{2}(15.0 + 25.1)(123.3 - 77.6) = 916.285$$

$$A_5 = \frac{1}{2}(15.1 + 15.0)(131.0 - 123.3) = 115.885$$

$$A_6 = \frac{1}{2}(15.1)(150.0 - 131.0) = 143.45$$

So, the total area can be estimated by

$$\begin{aligned} A &= A_1 + A_2 + A_3 + A_4 + A_5 + A_6 \\ &= 2799.34. \end{aligned}$$

c. **Suppose that the speed of the water was measured at $124 \frac{\text{ft.}}{\text{min}}$. What is the approximate volumetric flow in this section of the river, measured in gallons per minute?**

The volumetric flow is approximately $2799.34\ \text{ft}^2 \left(124 \frac{\text{ft.}}{\text{min}}\right) \approx 347,118 \frac{\text{ft}^3}{\text{min}}$. *Converting to gallons per minute, this is*

$$\left(347\,118 \frac{\text{ft}^3}{\text{min}}\right)\left(7.48052 \frac{\text{gallons}}{\text{ft}^3}\right) \approx 2\,596\,623\ \text{gallons/min}.$$