



## Lesson 3: Rational Exponents—What are $2^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ ?

### Student Outcomes

- Students calculate quantities that involve positive and negative rational exponents.

### Lesson Notes

Students extend their understanding of integer exponents to rational exponents by examining the graph of  $f(x) = 2^x$  and estimating the values of  $2^{\frac{1}{2}}$  and  $2^{\frac{1}{3}}$ . The lesson establishes the meaning of these numbers in terms of radical expressions, and these form the basis of how expressions of the form  $b^{\frac{1}{n}}$  are defined before generalizing further to expressions of the form  $b^{\frac{m}{n}}$ , where  $b$  is a positive real number and  $m$  and  $n$  are integers, with  $n \neq 0$  (**N-RN.A.1**). The lesson and Problem Set provide fluency practice in applying the properties of exponents to expressions containing rational exponents and radicals (**N-RN.A.2**). In the following lesson, students verify that the definition of an expression with rational exponents,  $b^{\frac{m}{n}} = \sqrt[n]{b^m}$ , is consistent with the remaining exponential properties. The lesson begins with students creating a graph of a simple exponential function,  $f(x) = 2^x$ , which they studied in Module 3 of Algebra I. In that module, students also learned about geometric sequences and their relationship to exponential functions, which is a concept that is revisited at the end of this module. In Algebra I, students worked with exponential functions with integer domains. This lesson, together with the subsequent Lessons 4 and 5, helps students understand why the domain of an exponential function  $f(x) = b^x$ , where  $b$  is a positive number and  $b \neq 1$ , is the set of real numbers. To do so, it is necessary to establish what it means to raise  $b$  to a rational power and, in Lesson 5, to any real number power.

### Classwork

#### Opening (1 minute)

In Algebra I, students worked with geometric sequences and simple exponential functions. Remind them that in Lesson 1, they created formulas based on repeatedly doubling and halving a number when they modeled folding a piece of paper. They reviewed how to use the properties of exponents for expressions that had integer exponents.

#### Opening Exercise (5 minutes)

Have students graph  $f(x) = 2^x$  for each integer  $x$  from  $x = -2$  to  $x = 5$  on the axes provided without using a graphing utility or calculator. Discuss the pattern of points, and ask students to connect the points in a way that produces a smooth curve. Students should work these two problems independently. If time permits, have them check their solutions with a partner before leading a whole-class discussion to review the solutions.

#### Scaffolding:

- Encourage students to create a table of values to help them construct the graph.
- For students working above grade level, have them repeat these exercises with the function  $f(x) = 3^x$ , and ask them to estimate  $3^{\frac{1}{2}}$  and  $3^{\frac{1}{3}}$ .

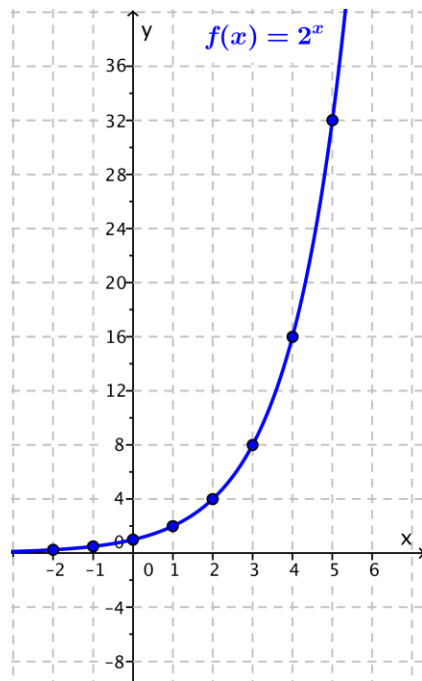
MP.3

Opening Exercise

a. What is the value of  $2^{\frac{1}{2}}$ ? Justify your answer.

*A possible student response follows: I think it will be around 1.5 because  $2^0 = 1$  and  $2^1 = 2$ .*

b. Graph  $f(x) = 2^x$  for each integer  $x$  from  $x = -2$  to  $x = 5$ . Connect the points on your graph with a smooth curve.



Ask for a few volunteers to explain their reasoning for their answers to Opening Exercise, part (a). Then, debrief these two exercises by leading a short discussion.

- The directions in the Opening Exercise said to connect the points with a smooth curve. What does it imply about the domain of a function when we connect points that we have plotted?
  - *That the domain of the function includes the points between the points that we plotted.*
- How does your graph support or refute your answer to the first exercise? Do you need to modify your answer to the question: What is the value of  $2^{\frac{1}{2}}$ ? Why or why not?
  - *If the domain is all real numbers, then the value of  $2^{\frac{1}{2}}$  will be the y-coordinate of the point on the graph where  $x = \frac{1}{2}$ . From the graph, it looks like the value is somewhere between 1 and 2. The scaling on this graph is not detailed enough for me to accurately refine my answer yet.*

Transition to the next set of exercises by telling students that they can better estimate the value of  $2^{\frac{1}{2}}$  by looking at a graph that is scaled more precisely.

The graph of  $f(x) = 2^x$  shown below for the next exercises appears in the student materials, but the graph could also be displayed using technology. Have students estimate the value of  $2^{\frac{1}{2}}$  and  $2^{\frac{1}{3}}$  from the graph. Students should work by themselves or in pairs to work through the following questions without using a calculator.

The graph on the right shows a close-up view of  $f(x) = 2^x$  for  $-0.5 < x < 1.5$ .

c. Find two consecutive integers that are under and over estimates of the value of  $2^{\frac{1}{2}}$ .

$2^0 < 2^{\frac{1}{2}} < 2^1$   
 $1 < 2^{\frac{1}{2}} < 2$

d. Does it appear that  $2^{\frac{1}{2}}$  is halfway between the integers you specified in Exercise 1?

No. It looks like  $2^{\frac{1}{2}}$  is a little less than halfway between 1 and 2.

e. Use the graph of  $f(x) = 2^x$  to estimate the value of  $2^{\frac{1}{2}}$ .

$2^{\frac{1}{2}} \approx 1.4$

f. Use the graph of  $f(x) = 2^x$  to estimate the value of  $2^{\frac{1}{3}}$ .

$2^{\frac{1}{3}} \approx 1.25$

**Discussion (9 minutes)**

Before getting into the point of this lesson, which is to connect rational exponents to radical expressions, revisit the initial question with students.

MP.6

- What is the value of  $2^{\frac{1}{2}}$ ? Does anyone want to adjust his or her initial guess?
  - Our initial guess was a little too big. It seems like 1.4 might be a better answer.
- How could we make a better guess?
  - We could look at the graph with even smaller increments for the scale using technology.

*Scaffolding:*

If needed, demonstrate the argument using perfect squares. For example, use a base of 4 instead of a base of 2.

Show that  $(4^{\frac{1}{2}})^2 = 4$   
 and  $(\sqrt{4})^2 = 4$ .

If time permits, zoom in further on the graph of  $f(x) = 2^x$  using a graphing calculator or other technology either by examining a graph or a table of values of  $x$  closer and closer to  $\frac{1}{2}$ .

Next, make the connection that  $2^{\frac{1}{2}} = \sqrt{2}$ . Walk students through the following questions, providing guidance as needed. Students proved that there was only one positive number that squared to 2 in Geometry, Module 2. It may be necessary to remind them of this with a bit more detail if they are struggling to follow this argument.

MP.7

- Assume for the moment that whatever  $2^{\frac{1}{2}}$  means, it satisfies our known rule for integer exponents  $b^m \cdot b^n = b^{m+n}$ . Working with this assumption, what is the value of  $2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}}$ ?
  - *It would be 2 because  $2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 2^{\frac{1}{2}+\frac{1}{2}} = 2^1 = 2$ .*
- What unique positive number squares to 2? That is, what is the only positive number that when multiplied by itself is equal to 2?
  - *By definition, we call the unique positive number that squares to 2 the square root of 2, and we write  $\sqrt{2}$ .*

Write the following statements on the board, and ask students to compare them and think about what the statements must tell them about the meaning of  $2^{\frac{1}{2}}$ .

$$2^{\frac{1}{2}} \cdot 2^{\frac{1}{2}} = 2 \text{ and } \sqrt{2} \cdot \sqrt{2} = 2$$

- What do these two statements tell us about the meaning of  $2^{\frac{1}{2}}$ ?
  - *Since both statements involve multiplying a number by itself and getting 2, and we know that there is only one number that does that, we can conclude that  $2^{\frac{1}{2}} = \sqrt{2}$ .*

At this point, have students confirm these results by using a calculator to approximate both  $2^{\frac{1}{2}}$  and  $\sqrt{2}$  to several decimal places. In the Opening,  $2^{\frac{1}{2}}$  was approximated graphically, and now it has been shown to be an irrational number.

Next, ask students to think about the meaning of  $2^{\frac{1}{3}}$  using a similar line of reasoning.

MP.7

- Assume that whatever  $2^{\frac{1}{3}}$  means will satisfy  $b^m \cdot b^n = b^{m+n}$ . What is the value of  $(2^{\frac{1}{3}})(2^{\frac{1}{3}})(2^{\frac{1}{3}})$ ?
  - *The value is 2 because  $(2^{\frac{1}{3}})(2^{\frac{1}{3}})(2^{\frac{1}{3}}) = 2^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = 2^1 = 2$ .*
- What is the value of  $\sqrt[3]{2} \cdot \sqrt[3]{2} \cdot \sqrt[3]{2}$ ?
  - *The value is 2 because  $\sqrt[3]{2} \cdot \sqrt[3]{2} \cdot \sqrt[3]{2} = (\sqrt[3]{2})^3 = 2$ .*
- What appears to be the meaning of  $2^{\frac{1}{3}}$ ?
  - *Since both the exponent expression and the radical expression involve multiplying a number by itself three times and the result is equal to 2, we know that  $2^{\frac{1}{3}} = \sqrt[3]{2}$ .*

**Scaffolding:**

If needed, demonstrate the argument using perfect cubes. For example, use a base of 8 instead of a base of 2.

$$\text{Show } \left(8^{\frac{1}{3}}\right)^3 = 8$$

$$\text{and } \left(\sqrt[3]{8}\right)^3 = 8.$$

Students can also confirm using a calculator that the decimal approximations of  $2^{\frac{1}{3}}$  and  $\sqrt[3]{2}$  are the same. Next, they are asked to generalize their findings.

- Can we generalize this relationship? Does  $2^{\frac{1}{4}} = \sqrt[4]{2}$ ? Does  $2^{\frac{1}{10}} = \sqrt[10]{2}$ ? What is  $2^{\frac{1}{n}}$ , for any positive integer  $n$ ? Why?

- $2^{\frac{1}{n}} = \sqrt[n]{2}$  because

$$\left(2^{\frac{1}{n}}\right)^n = \underbrace{\left(2^{\frac{1}{n}}\right)\left(2^{\frac{1}{n}}\right)\cdots\left(2^{\frac{1}{n}}\right)}_{n \text{ times}} = \overbrace{2^{\frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}}^{n \text{ times}} = 2^1 = 2.$$

MP.8

Have students confirm these results using a calculator as well as checking to see if the decimal approximations of  $2^{\frac{1}{n}}$  and  $\sqrt[n]{2}$  are the same for different values of  $n$  such as 4, 5, 6, 10, .... Be sure to share the generalization shown above on the board to help students understand why it makes sense to define  $2^{\frac{1}{n}}$  to be  $\sqrt[n]{2}$ .

However, be careful to not stop here; there is a problem with the reasoning if  $\sqrt[n]{2}$  is not defined. In previous courses, only square roots and cube roots were defined.

It is first necessary to define the  $n^{\text{th}}$  root of a number; there may be more than one, as in the case where  $2^2 = 4$  and  $(-2)^2 = 4$ . It is said that both  $-2$  and  $2$  are square roots of 4. However, priority is given to the positive-valued square root, and it is said that  $2$  is the *principal square root* of 4. Often *the* square root of 4 is referred to when what is meant is the *principal* square root of 4. The definition of  $n^{\text{th}}$  root presented below is consistent with allowing complex  $n^{\text{th}}$  roots, which students encounter in Precalculus and in college if they pursue engineering or higher mathematics. If complex  $n^{\text{th}}$  roots are allowed, there are three cube roots of 2:  $\sqrt[3]{2}$ ,  $\sqrt[3]{2}\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$ , and  $\sqrt[3]{2}\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$ , and the real number  $\sqrt[3]{2}$  is referred to as the *principal cube root* of 2. There is no need to discuss this with any but the most advanced students.

The  $n^{\text{th}}$  root of 2,  $\sqrt[n]{2}$ , is the positive real number  $a$  such that  $a^n = 2$ . In general, if  $a$  is positive, then the  $n^{\text{th}}$  root of  $a$  exists for any positive integer  $n$ , and if  $a$  is negative, then the  $n^{\text{th}}$  root of  $a$  exists only for odd integers  $n$ . This even/odd condition is handled subtly in the definition below; the  $n^{\text{th}}$  root exists only if there is already an exponential relationship  $b = a^n$ .

Present the following definitions to students, and have them record them in their notes.

**$n^{\text{th}}$  ROOT OF A NUMBER:** Let  $a$  and  $b$  be numbers, and let  $n$  be a positive integer. If  $b = a^n$ , then  $a$  is a  $n^{\text{th}}$  root of  $b$ . If  $n = 2$ , then the root is called a *square root*. If  $n = 3$ , then the root is called a *cube root*.

**PRINCIPAL  $n^{\text{th}}$  ROOT OF A NUMBER:** Let  $b$  be a real number that has at least one real  $n^{\text{th}}$  root. The *principal  $n^{\text{th}}$  root of  $b$*  is the real  $n^{\text{th}}$  root that has the same sign as  $b$  and is denoted by a radical symbol:  $\sqrt[n]{b}$ .

Every positive number has a unique principal  $n^{\text{th}}$  root. We often refer to the principal  $n^{\text{th}}$  root of  $b$  as just the  $n^{\text{th}}$  root of  $b$ . For any positive integer  $n$ , the  $n^{\text{th}}$  root of 0 is 0.

Students have already learned about square and cube roots in previous grades. In Module 1 and at the beginning of this lesson, students worked with radical expressions involving cube and square roots. Explain that the  $n^{\text{th}}$  roots of a number satisfy the same properties of radicals learned previously. Have students record these properties in their notes.

If  $a \geq 0$ ,  $b \geq 0$  ( $b \neq 0$  when  $b$  is a denominator), and  $n$  is a positive integer, then

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b} \quad \text{and} \quad \sqrt[n]{\frac{a}{b}} = \frac{\sqrt[n]{a}}{\sqrt[n]{b}}.$$

Background information regarding  $n^{\text{th}}$  roots and their uniqueness is provided below. Consider sharing this with students working above grade level or the entire class if extending this lesson to an additional day.

The existence of the principal  $n^{\text{th}}$  root of a positive real number  $b$  is a consequence of the fundamental theorem of algebra: Consider the polynomial function  $f(x) = x^n - b$ . When  $n$  is odd, we know that  $f$  has at least one real zero because the graph of  $f$  must cross the  $x$ -axis. That zero is a positive number, which (after showing that it is the only real zero) is the  $n^{\text{th}}$  root. The case for when  $n$  is even follows a similar argument.

To show uniqueness of the  $n^{\text{th}}$  root, suppose there are two  $n^{\text{th}}$  roots of a number  $b$ ,  $x$ , and  $y$ , such that  $x > 0$ ,  $y > 0$ ,  $x^n = b$ , and  $y^n = b$ . Then,  $x^n - y^n = b - b = 0$ , and the expression  $x^n - y^n$  factors (see Lesson 7 in Module 1).

$$0 = x^n - y^n$$

$$0 = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \dots + xy^{n-2} + y^{n-1})$$

Since both  $x$  and  $y$  are positive, the second factor is never zero. Thus, for  $x^n - y^n = 0$ , we must have  $x - y = 0$ , and it follows that  $x = y$ . Thus, there is only one  $n^{\text{th}}$  root of  $b$ .

A proof of the first radical property is shown below for background information. Consider sharing this proof with students working above grade level or the entire class if extending this lesson to an additional day.

Prove that  $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$ .

Let  $x \geq 0$  be the number such that  $x^n = a$ , and let  $y \geq 0$  be the number such that  $y^n = b$ , so that  $x = \sqrt[n]{a}$  and  $y = \sqrt[n]{b}$ . Then, by a property of exponents,  $(xy)^n = x^n y^n = ab$ . Thus,  $xy$  must be the  $n^{\text{th}}$  root of  $ab$ . Writing this using our notation gives

$$\sqrt[n]{ab} = xy = \sqrt[n]{a} \cdot \sqrt[n]{b}.$$

**Example (3 minutes)**

This example familiarizes students with the wording in the definition presented above.

**Example**

a. What is the 4<sup>th</sup> root of 16?  
 $x^4 = 16$  when  $x = 2$  because  $2^4 = 16$ . Thus,  $\sqrt[4]{16} = 2$ .

b. What is the cube root of 125?  
 $x^3 = 125$  when  $x = 5$  because  $5^3 = 125$ . Thus,  $\sqrt[3]{125} = 5$ .

c. What is the 5<sup>th</sup> root of 100,000?  
 $x^5 = 100,000$  when  $x = 10$  because  $10^5 = 100,000$ . Thus,  $\sqrt[5]{100,000} = 10$ .

**Exercise 1 (2 minutes)**

In these brief exercises, students work with the definition of  $n^{\text{th}}$  roots and the multiplication property presented above. Have students check their work with a partner and briefly discuss any questions that arise as a whole class.

**Exercise 1**

1. Evaluate each expression.

a.  $\sqrt[4]{81}$   
 3

b.  $\sqrt[5]{32}$   
 2

c.  $\sqrt[3]{9} \cdot \sqrt[3]{3}$   
 $\sqrt[3]{27} = 3$

d.  $\sqrt[4]{25} \cdot \sqrt[4]{100} \cdot \sqrt[4]{4}$   
 $\sqrt[4]{10,000} = 10$

*Scaffolding:*  
 If needed, continue to support students who struggle with abstraction by including additional numeric examples.

**Discussion (8 minutes)**

Return to the question posted in the title of this lesson: What are  $2^{\frac{1}{2}}$  and  $2^{\frac{1}{3}}$ ? Now we know the answer,  $2^{\frac{1}{2}} = \sqrt{2}$  and  $2^{\frac{1}{3}} = \sqrt[3]{2}$ . So far, we have given meaning to  $2^{\frac{1}{n}}$  by equating  $2^{\frac{1}{n}}$  and  $\sqrt[n]{2}$ . Ask students if they believe these results extend to any base  $b > 0$ .

- We just did some exercises where the  $b$ -value was a number different from 2. In our earlier work, was there something special about using 2 as the base of our exponential expression? Would these results generalize to expressions of the form  $3^{\frac{1}{n}}$ ?  $7^{\frac{1}{n}}$ ?  $10^{\frac{1}{n}}$ ?  $b^{\frac{1}{n}}$ , for any positive real number  $b$ ?
  - *There is nothing inherently special about the base 2 in the above discussion. These results should generalize to expressions of the form  $b^{\frac{1}{n}}$  for any positive real number base  $b$  because we have defined an  $n^{\text{th}}$  root for any positive base  $b$ .*

Now that we know the meaning of the  $n^{\text{th}}$  root of a number  $b$ ,  $\sqrt[n]{b}$ , the work done earlier with base 2 suggests that  $b^{\frac{1}{n}}$  should also be defined as the  $n^{\text{th}}$  root of  $b$ . Discuss the definition below with the class. If students are unclear on the definition, do some numerical examples. For example,  $(-32)^{\frac{1}{5}} = -2$  because  $(-2)^5 = -32$ , but  $(-16)^{\frac{1}{4}}$  does not exist because there is no principal 4<sup>th</sup> root of a negative number.

For a real number  $b$  and a positive integer  $n$ , define  $b^{\frac{1}{n}}$  to be the principal  $n^{\text{th}}$  root of  $b$  when it exists. That is,

$$b^{\frac{1}{n}} = \sqrt[n]{b}.$$

Note that the definition of  $\sqrt[n]{b}$  holds for any real number  $b$  if  $n$  is an odd integer and for positive real numbers  $b$  if  $n$  is an even integer. Consider emphasizing this with the class. Thus, when  $b$  is negative and  $n$  is an odd integer, the expression  $b^{\frac{1}{n}}$  is negative. If  $n$  is an even integer, then  $b$  must be restricted to positive real numbers only, and  $b^{\frac{1}{n}}$  is positive.

In the next lesson, students see that with this definition,  $b^{\frac{1}{n}}$  satisfies all the usual properties of exponents, so it makes sense to define it in this way.

At this point, consider revisiting the original question with students one more time.

- What is the value of  $2^{\frac{1}{2}}$ ? What does it mean? What is the value of  $b^{\frac{1}{2}}$  for any positive number  $b$ ? How are radicals related to rational exponents?
  - *We estimated numerically that  $2^{\frac{1}{2}} \approx 1.4$  in part (e) of the Opening Exercise. We now know that  $2^{\frac{1}{2}}$  is equal to  $\sqrt{2}$ . In general,  $b^{\frac{1}{2}} = \sqrt{b}$  for any positive real number  $b$ . A number that contains a radical can be expressed using rational exponents in place of the radical.*

In the following Discussion, the definition of exponentiation with exponents of the form  $\frac{1}{n}$  is extended to exponentiation with any positive rational exponent. Begin by considering an example: What do we mean by  $2^{\frac{3}{4}}$ ? Give students a few minutes to respond individually in writing to this question on their student pages, and then have them discuss their reasoning with a partner. Make sure to correct any blatant misconceptions and to clarify incomplete thinking while leading the Discussion that follows.

**Discussion**

If  $2^{\frac{1}{2}} = \sqrt{2}$  and  $2^{\frac{1}{3}} = \sqrt[3]{2}$ , what does  $2^{\frac{3}{4}}$  equal? Explain your reasoning.

*Student solutions and explanations will vary. One possible solution would be  $2^{\frac{3}{4}} = \left(\frac{1}{2^4}\right)^3$ , so it must mean that  $2^{\frac{3}{4}} = \left(\sqrt[4]{2}\right)^3$ . Since the properties of exponents and the meaning of an exponent made sense with integers and now for rational numbers in the form  $\frac{1}{n}$ , it would make sense that they would work for all rational numbers, too.*

MP.3

Now that there is a definition for exponential expressions of the form  $b^{\frac{1}{n}}$ , use the discussion below to define  $b^{\frac{m}{n}}$ , where  $m$  and  $n$  are integers,  $n \neq 0$ , and  $b$  is a positive real number. Make sure students understand that the interpretation of  $b^{\frac{m}{n}}$  must be consistent with the exponent properties (which hold for integer exponents) and the definition of  $b^{\frac{1}{n}}$ .

- How can we rewrite the exponent of  $2^{\frac{3}{4}}$  using integers and rational numbers in the form  $\frac{1}{n}$ ?
  - *We can write  $2^{\frac{3}{4}} = \left(\frac{1}{2^4}\right)^3$ , or we can write  $2^{\frac{3}{4}} = (2^3)^{\frac{1}{4}}$ .*



- Now, apply our definition of  $b^{\frac{1}{n}}$ .
  - $2^{\frac{3}{4}} = \left(2^{\frac{1}{4}}\right)^3 = \left(\sqrt[4]{2}\right)^3$  or  $2^{\frac{3}{4}} = (2^3)^{\frac{1}{4}} = \sqrt[4]{2^3} = \sqrt[4]{8}$
- Does this make sense? If  $2^{\frac{3}{4}} = \sqrt[4]{8}$ , then if we raise  $2^{\frac{3}{4}}$  to the fourth power, we should get 8. Does this happen?
  - $\left(2^{\frac{3}{4}}\right)^4 = \left(2^{\frac{3}{4}}\right)\left(2^{\frac{3}{4}}\right)\left(2^{\frac{3}{4}}\right)\left(2^{\frac{3}{4}}\right) = 2^{\left(4 \cdot \frac{3}{4}\right)} = 2^3 = 8$
  - So, 8 is the product of four equal factors, which we denote by  $2^{\frac{3}{4}}$ . Thus,  $2^{\frac{3}{4}} = \sqrt[4]{8}$ .

Take a few minutes to allow students to think about generalizing their work above to  $2^{\frac{m}{n}}$  and then to  $b^{\frac{m}{n}}$ . Have them write a response to the following questions and share it with a partner before proceeding as a whole class.

- Can we generalize this result? How would you define  $2^{\frac{m}{n}}$ , for positive integers  $m$  and  $n$ ?
  - Conjecture:  $2^{\frac{m}{n}} = \sqrt[n]{2^m}$ , or equivalently,  $2^{\frac{m}{n}} = \left(\sqrt[n]{2}\right)^m$ .
- Can we generalize this result to any positive real number base  $b$ ? What is  $3^{\frac{m}{n}}$ ?  $7^{\frac{m}{n}}$ ?  $10^{\frac{m}{n}}$ ?  $b^{\frac{m}{n}}$ ?
  - There is nothing inherently special about the base 2 in the above Discussion. These results should generalize to expressions of the form  $b^{\frac{m}{n}}$  for any positive real number base  $b$ .
  - Then we are ready to define  $b^{\frac{m}{n}} = \sqrt[n]{b^m}$  for positive integers  $m$  and  $n$  and positive real numbers  $b$ .

This result is summarized in the box below.

For any positive integers  $m$  and  $n$ , and any real number  $b$  for which  $b^{\frac{1}{n}}$  exists, we define

$$b^{\frac{m}{n}} = \sqrt[n]{b^m}, \text{ which is equivalent to } b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m.$$

Note that this property holds for any real number  $b$  if  $n$  is an odd integer. Consider emphasizing this with the class. When  $b$  is negative and  $n$  is an odd integer, the expression  $b^{\frac{m}{n}}$  is negative. If  $n$  is an even integer, then  $b$  must be restricted to positive real numbers only.

**Exercises 2–8 (4 minutes)**

In these exercises, students use the definitions above to rewrite and evaluate expressions. Have students check their work with a partner and briefly discuss as a whole class any questions that arise.

**Exercises 2–12**  
 Rewrite each exponential expression as a radical expression.

2.  $3^{\frac{1}{2}}$   
 $3^{\frac{1}{2}} = \sqrt{3}$

3.  $11^{\frac{1}{5}}$

$$11^{\frac{1}{5}} = \sqrt[5]{11}$$

4.  $\left(\frac{1}{4}\right)^{\frac{1}{5}}$

$$\left(\frac{1}{4}\right)^{\frac{1}{5}} = \sqrt[5]{\frac{1}{4}}$$

5.  $6^{\frac{1}{10}}$

$$6^{\frac{1}{10}} = \sqrt[10]{6}$$

Rewrite the following exponential expressions as equivalent radical expressions. If the number is rational, write it without radicals or exponents.

6.  $2^{\frac{3}{2}}$

$$2^{\frac{3}{2}} = \sqrt{2^3} = 2\sqrt{2}$$

7.  $4^{\frac{5}{2}}$

$$4^{\frac{5}{2}} = \sqrt{4^5} = (\sqrt{4})^5 = 2^5 = 32$$

8.  $\left(\frac{1}{8}\right)^{\frac{5}{3}}$

$$\left(\frac{1}{8}\right)^{\frac{5}{3}} = \sqrt[3]{\left(\frac{1}{8}\right)^5} = \left(\sqrt[3]{\frac{1}{8}}\right)^5 = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$$

### Exercise 9 (3 minutes)

In this exercise, students are asked to consider a negative rational exponent. Have students work directly with a partner, and ask them to use thinking similar to that in the preceding Discussion. Correct and extend student thinking while reviewing the solution.

9. Show why the following statement is true:

$$2^{-\frac{1}{2}} = \frac{1}{2^{\frac{1}{2}}}$$

*Student solutions and explanations will vary. One possible solution would be*

$$2^{-\frac{1}{2}} = \left(2^{\frac{1}{2}}\right)^{-1} = (\sqrt{2})^{-1} = \frac{1}{\sqrt{2}} = \frac{1}{2^{\frac{1}{2}}}$$

Share the following property with the class, and show how the work they did in the previous exercises supports this conclusion. Consider verifying these properties using an argument similar to the ones presented earlier for the meaning of  $b^{\frac{m}{n}}$ .

For any positive integers  $m$  and  $n$ , and any nonzero number  $b$  for which  $b^{\frac{1}{n}}$  exists, we define

$$b^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{b^m}}$$

or, equivalently,

$$b^{-\frac{m}{n}} = \frac{1}{(\sqrt[n]{b})^m}.$$

### Exercises 10–12 (3 minutes)

Rewrite the following exponential expressions as equivalent radical expressions. If the number is rational, write it without radicals or exponents.

10.  $4^{-\frac{3}{2}}$

$$4^{-\frac{3}{2}} = \frac{1}{\sqrt{4^3}} = \frac{1}{(\sqrt{4})^3} = \frac{1}{8}$$

11.  $27^{-\frac{2}{3}}$

$$27^{-\frac{2}{3}} = \frac{1}{27^{\frac{2}{3}}} = \frac{1}{(\sqrt[3]{27})^2} = \frac{1}{3^2} = \frac{1}{9}$$

12.  $\left(\frac{1}{4}\right)^{-\frac{1}{2}}$

We have  $\left(\frac{1}{4}\right)^{-\frac{1}{2}} = \left(\sqrt{\frac{1}{4}}\right)^{-1} = \left(\frac{1}{2}\right)^{-1} = 2$ . Alternatively,  $\left(\frac{1}{4}\right)^{-\frac{1}{2}} = \left(\left(\frac{1}{4}\right)^{-1}\right)^{\frac{1}{2}} = (4)^{\frac{1}{2}} = \sqrt{4} = 2$ .

### Closing (3 minutes)

Have students summarize the key points of the lesson in writing. Circulate around the classroom to informally assess understanding and provide assistance. Student work should reflect the summary provided below.

## Lesson Summary

**$n^{\text{th}}$  ROOT OF A NUMBER:** Let  $a$  and  $b$  be numbers, and let  $n$  be a positive integer. If  $b = a^n$ , then  $a$  is a  $n^{\text{th}}$  root of  $b$ . If  $n = 2$ , then the root is called a *square root*. If  $n = 3$ , then the root is called a *cube root*.

**PRINCIPAL  $n^{\text{th}}$  ROOT OF A NUMBER:** Let  $b$  be a real number that has at least one real  $n^{\text{th}}$  root. The *principal  $n^{\text{th}}$  root of  $b$*  is the real  $n^{\text{th}}$  root that has the same sign as  $b$  and is denoted by a radical symbol:  $\sqrt[n]{b}$ .

Every positive number has a unique principal  $n^{\text{th}}$  root. We often refer to the principal  $n^{\text{th}}$  root of  $b$  as just *the  $n^{\text{th}}$  root of  $b$* . The  $n^{\text{th}}$  root of 0 is 0.

For any positive integers  $m$  and  $n$ , and any real number  $b$  for which the principal  $n^{\text{th}}$  root of  $b$  exists, we have

$$b^{\frac{1}{n}} = \sqrt[n]{b}$$

$$b^{\frac{m}{n}} = \sqrt[n]{b^m} = (\sqrt[n]{b})^m$$

$$b^{-\frac{m}{n}} = \frac{1}{\sqrt[n]{b^m}} \text{ for } b \neq 0.$$

## Exit Ticket (4 minutes)

Name \_\_\_\_\_

Date \_\_\_\_\_

## Lesson 3: Rational Exponents—What are $2^{\frac{1}{2}}$ and $2^{\frac{1}{3}}$ ?

### Exit Ticket

- Write the following exponential expressions as equivalent radical expressions.
  - $2^{\frac{1}{2}}$
  - $2^{\frac{3}{4}}$
  - $3^{-\frac{2}{3}}$
- Rewrite the following radical expressions as equivalent exponential expressions.
  - $\sqrt{5}$
  - $2^4\sqrt{3}$
  - $\frac{1}{\sqrt[3]{16}}$
- Provide a written explanation for each question below.
  - Is it true that  $\left(1000^{\frac{1}{3}}\right)^3 = (1000^3)^{\frac{1}{3}}$ ? Explain how you know.
  - Is it true that  $\left(4^{\frac{1}{2}}\right)^3 = (4^3)^{\frac{1}{2}}$ ? Explain how you know.
  - Suppose that  $m$  and  $n$  are positive integers and  $b$  is a real number so that the principal  $n^{\text{th}}$  root of  $b$  exists. In general, does  $\left(b^{\frac{1}{n}}\right)^m = (b^m)^{\frac{1}{n}}$ ? Explain how you know.

## Exit Ticket Sample Solutions

1. Rewrite the following exponential expressions as equivalent radical expressions.

a.  $2^{\frac{1}{2}}$

$$2^{\frac{1}{2}} = \sqrt{2}$$

b.  $2^{\frac{3}{4}}$

$$2^{\frac{3}{4}} = \sqrt[4]{2^3} = \sqrt[4]{8}$$

c.  $3^{-\frac{2}{3}}$

$$3^{-\frac{2}{3}} = \frac{1}{\sqrt[3]{3^2}} = \frac{1}{\sqrt[3]{9}}$$

2. Rewrite the following radical expressions as equivalent exponential expressions.

a.  $\sqrt{5}$

$$\sqrt{5} = 5^{\frac{1}{2}}$$

b.  $2^4\sqrt{3}$

$$2^4\sqrt{3} = \sqrt[4]{2^4 \cdot 3} = \sqrt[4]{48} = 48^{\frac{1}{4}}$$

c.  $\frac{1}{\sqrt[3]{16}}$

$$\frac{1}{\sqrt[3]{16}} = (2^4)^{-\frac{1}{3}} = 2^{-\frac{4}{3}}$$

$$\frac{1}{\sqrt[3]{16}} = (16)^{-\frac{1}{3}}$$

3. Provide a written explanation for each question below.

a. Is it true that  $(1000^{\frac{1}{3}})^3 = (1000^3)^{\frac{1}{3}}$ ? Explain how you know.

$$(1000^{\frac{1}{3}})^3 = (\sqrt[3]{1000})^3 = 10^3 = 1000$$

$$(1000^3)^{\frac{1}{3}} = (1000000000)^{\frac{1}{3}} = 1000$$

*So, this statement is true.*

b. Is it true that  $(4^{\frac{1}{2}})^3 = (4^3)^{\frac{1}{2}}$ ? Explain how you know.

$$(4^{\frac{1}{2}})^3 = (\sqrt{4})^3 = 2^3 = 8$$

$$(4^3)^{\frac{1}{2}} = 64^{\frac{1}{2}} = \sqrt{64} = 8$$

*So, this statement is true.*

MP.3  
&  
MP.7

- c. Suppose that  $m$  and  $n$  are positive integers and  $b$  is a real number so that the principal  $n^{\text{th}}$  root of  $b$  exists. In general, does  $\left(b^{\frac{1}{n}}\right)^m = (b^m)^{\frac{1}{n}}$ ? Explain how you know.

*From the two examples we have seen, it appears that we can extend the property  $\left(b^{\frac{1}{n}}\right)^m = (b^m)^{\frac{1}{n}}$  for integers  $m$  and  $n$  to rational exponents.*

*We know that, in general, we have*

$$\begin{aligned} \left(b^{\frac{1}{n}}\right)^m &= \left(\sqrt[n]{b}\right)^m \\ &= \sqrt[n]{b^m} \\ &= (b^m)^{\frac{1}{n}}. \end{aligned}$$

Problem Set Sample Solutions

1. Select the expression from (A), (B), and (C) that correctly completes the statement.

a.  $x^{\frac{1}{3}}$  is equivalent to \_\_\_\_\_.      (A)  $\frac{1}{3}x$       (B)  $\sqrt[3]{x}$       (C)  $\frac{3}{x}$

(B)

b.  $x^{\frac{2}{3}}$  is equivalent to \_\_\_\_\_.      (A)  $\frac{2}{3}x$       (B)  $\sqrt[3]{x^2}$       (C)  $(\sqrt{x})^3$

(B)

c.  $x^{-\frac{1}{4}}$  is equivalent to \_\_\_\_\_.      (A)  $-\frac{1}{4}x$       (B)  $\frac{4}{x}$       (C)  $\frac{1}{\sqrt[4]{x}}$

(C)

d.  $\left(\frac{4}{x}\right)^{\frac{1}{2}}$  is equivalent to \_\_\_\_\_.      (A)  $\frac{2}{x}$       (B)  $\frac{4}{x^2}$       (C)  $\frac{2}{\sqrt{x}}$

(C)

2. Identify which of the expressions (A), (B), and (C) are equivalent to the given expression.

a.  $16^{\frac{1}{2}}$       (A)  $\left(\frac{1}{16}\right)^{\frac{1}{2}}$       (B)  $8^{\frac{2}{3}}$       (C)  $64^{\frac{3}{2}}$

(A) and (B)

b.  $\left(\frac{2}{3}\right)^{-1}$       (A)  $-\frac{3}{2}$       (B)  $\left(\frac{9}{4}\right)^{\frac{1}{2}}$       (C)  $\frac{27^{\frac{1}{3}}}{6}$

(B) only

3. Rewrite in radical form. If the number is rational, write it without using radicals.

a.  $6^{\frac{3}{2}}$   
 $\sqrt{216}$

b.  $\left(\frac{1}{2}\right)^{\frac{1}{4}}$   
 $\sqrt[4]{\frac{1}{2}}$

c.  $3(8)^{\frac{1}{3}}$   
 $3\sqrt[3]{8} = 6$

d.  $\left(\frac{64}{125}\right)^{-\frac{2}{3}}$   
 $\left(\sqrt[3]{\frac{125}{64}}\right)^2 = \frac{25}{16}$

e.  $81^{-\frac{1}{4}}$   
 $\frac{1}{\sqrt[4]{81}} = \frac{1}{3}$

4. Rewrite the following expressions in exponent form.

a.  $\sqrt{5}$   
 $5^{\frac{1}{2}}$

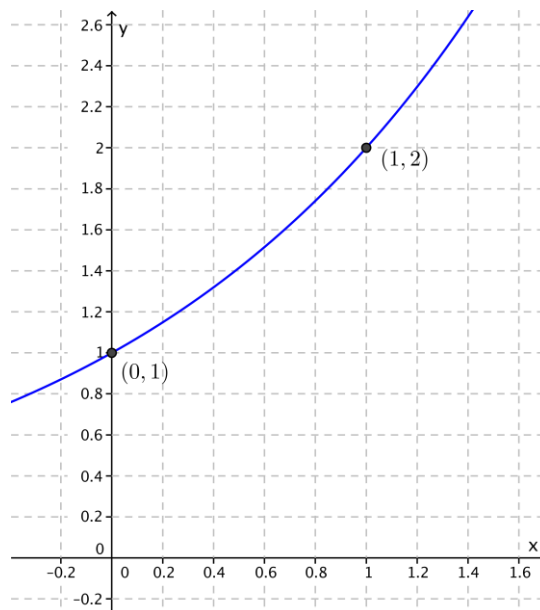
b.  $\sqrt[3]{5^2}$   
 $5^{\frac{2}{3}}$

c.  $\sqrt{5^3}$   
 $5^{\frac{3}{2}}$

d.  $(\sqrt[3]{5})^2$   
 $5^{\frac{2}{3}}$

5. Use the graph of  $f(x) = 2^x$  shown to the right to estimate the following powers of 2.

- a.  $2^{\frac{1}{4}} \approx 1.2$
- b.  $2^{\frac{2}{3}} \approx 1.6$
- c.  $2^{\frac{3}{4}} \approx 1.7$
- d.  $2^{0.2} \approx 1.15$
- e.  $2^{1.2} \approx 2.3$
- f.  $2^{-\frac{1}{5}} \approx 0.85$





6. Rewrite each expression in the form  $kx^n$ , where  $k$  is a real number,  $x$  is a positive real number, and  $n$  is rational.

a.  $\frac{\sqrt[4]{16x^3}}{2x^{\frac{3}{4}}}$

b.  $\frac{5}{\sqrt{x}} \cdot 5x^{\frac{1}{2}}$

c.  $\frac{\sqrt[3]{\quad}}{x^{\frac{4}{3}}}$

d.  $\frac{4}{\sqrt[3]{8x^3}} \cdot 2x^{-1}$

e.  $\frac{27}{\sqrt{9x^4}} \cdot 9x^{-2}$

f.  $\left(\frac{125}{x^2}\right)^{\frac{1}{3}} \cdot \frac{1}{5}x^{\frac{2}{3}}$

7. Find the value of  $x$  for which  $2x^{\frac{1}{2}} = 32$ .

256

8. Find the value of  $x$  for which  $x^{\frac{4}{3}} = 81$ .

27

9. If  $x^{\frac{3}{2}} = 64$ , find the value of  $4x^{-\frac{3}{4}}$ .

$$x = 16, \text{ so } 4(16)^{-\frac{3}{4}} = 4(2)^{-3} = \frac{4}{8} = \frac{1}{2}.$$

10. Evaluate the following expressions when  $b = \frac{1}{9}$ .

a.  $b^{-\frac{1}{2}}$   
 $\left(\frac{1}{9}\right)^{-\frac{1}{2}} = 9^{\frac{1}{2}} = 3$

b.  $b^{\frac{5}{2}}$   
 $\left(\frac{1}{9}\right)^{\frac{5}{2}} = \left(\frac{1}{3}\right)^5 = \frac{1}{243}$

c.  $\sqrt[3]{3b^{-1}}$   
 $\sqrt[3]{3\left(\frac{1}{9}\right)^{-1}} = \sqrt[3]{27} = 3$

11. Show that each expression is equivalent to  $2x$ . Assume  $x$  is a positive real number.

a.  $\sqrt[4]{16x^4}$   
 $\sqrt[4]{16} \cdot \sqrt[4]{x^4} = 2x$

b.  $\frac{(\sqrt[3]{8x^3})^2}{\sqrt{4x^2}}$   
 $\frac{(2x)^2}{(4x^2)^{\frac{1}{2}}} = \frac{4x^2}{2x} = 2x$

c.  $\frac{6x^3}{\sqrt[3]{27x^6}}$   
 $\frac{6x^3}{\frac{6}{3x^2}} = \frac{6x^3}{3x^2} = 2x$

12. Yoshiko said that  $16^{\frac{1}{4}} = 4$  because 4 is one-fourth of 16. Use properties of exponents to explain why she is or is not correct.

*Yoshiko's reasoning is not correct. By our exponent properties,  $(16^{\frac{1}{4}})^4 = 16^{(\frac{1}{4}) \cdot 4} = 16^1 = 16$ , but  $4^4 = 256$ .*

*Since  $(16^{\frac{1}{4}})^4 \neq 4^4$ , we know that  $16^{\frac{1}{4}} \neq 4$ .*

13. Jefferson said that  $8^{\frac{4}{3}} = 16$  because  $8^{\frac{1}{3}} = 2$  and  $2^4 = 16$ . Use properties of exponents to explain why he is or is not correct.

*Jefferson's reasoning is correct. We know that  $8^{\frac{4}{3}} = (8^{\frac{1}{3}})^4$ , so  $8^{\frac{4}{3}} = 2^4$ , and thus  $8^{\frac{4}{3}} = 16$ .*

14. Rita said that  $8^{\frac{2}{3}} = 128$  because  $8^{\frac{2}{3}} = 8^2 \cdot 8^{\frac{1}{3}}$ , so  $8^{\frac{2}{3}} = 64 \cdot 2$ , and then  $8^{\frac{2}{3}} = 128$ . Use properties of exponents to explain why she is or is not correct.

*Rita's reasoning is not correct because she did not apply the properties of exponents correctly. She should also realize that raising 8 to a positive power less than 1 would produce a number less than 8. The correct calculation is below.*

$$\begin{aligned} 8^{\frac{2}{3}} &= \left(8^{\frac{1}{3}}\right)^2 \\ &= 2^2 \\ &= 4 \end{aligned}$$

MP.3

15. Suppose for some positive real number  $a$  that  $\left(a^{\frac{1}{4}} \cdot a^{\frac{1}{2}} \cdot a^{\frac{1}{4}}\right)^2 = 3$ .

a. What is the value of  $a$ ?

$$\begin{aligned}\left(a^{\frac{1}{4}} \cdot a^{\frac{1}{2}} \cdot a^{\frac{1}{4}}\right)^2 &= 3 \\ \left(a^{\frac{1}{4} + \frac{1}{2} + \frac{1}{4}}\right)^2 &= 3 \\ (a^1)^2 &= 3 \\ a^2 &= 3 \\ a &= \sqrt{3}\end{aligned}$$

b. Which exponent properties did you use to find your answer to part (a)?

*We used the properties  $b^n \cdot b^m = b^{m+n}$  and  $(b^m)^n = b^{mn}$ .*

16. In the lesson, you made the following argument:

$$\begin{aligned}\left(2^{\frac{1}{3}}\right)^3 &= 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} \cdot 2^{\frac{1}{3}} \\ &= 2^{\frac{1}{3} + \frac{1}{3} + \frac{1}{3}} \\ &= 2^1 \\ &= 2.\end{aligned}$$

Since  $\sqrt[3]{2}$  is a number so that  $(\sqrt[3]{2})^3 = 2$  and  $2^{\frac{1}{3}}$  is a number so that  $(2^{\frac{1}{3}})^3 = 2$ , you concluded that  $2^{\frac{1}{3}} = \sqrt[3]{2}$ .

Which exponent property was used to make this argument?

*We used the property  $b^n \cdot b^m = b^{m+n}$ . (Students may also mention the uniqueness of  $n^{\text{th}}$  roots.)*