Lesson 6: Probability Rules

Student Outcomes

- Students use the complement rule to calculate the probability of the complement of an event and the multiplication rule for independent events to calculate the probability of the intersection of two independent events.
- Students recognize that two events A and B are independent if and only if $P(A \text{ and } B) = P(A)P(B)$ and interpret independence of two events A and B as meaning that the conditional probability of A given B is equal to $P(A)$.
- Students use the formula for conditional probability to calculate conditional probabilities and interpret probabilities in context.

Lesson Notes

This lesson introduces the formulas for calculating the probability of the complement of an event, the probability of an intersection when events are independent, and conditional probabilities. Because these concepts have already been presented using the more intuitive approach of earlier lessons, students should readily understand why the formulas are true.

Classwork

Opening (4 minutes)

Begin this lesson by asking students the following questions:

- There are 300 students at a certain school. All students indicated they were either right-handed or left-handed but not both. Fifty of the students are left-handed. How many students are right-handed? What is the probability of randomly selecting a right-handed student at this school? How did you determine this?
- The United States Census Bureau indicates there are $19{,}378{,}102$ people in the State of New York. If $4{,}324{,}929$ are under the age of 18, how many are 18 and older? What is the probability of randomly selecting a person from New York who is 18 or older? How did you determine this?
- It is estimated that approximately 20% of the people in the United States have asthma and severe allergies. What is the probability that a randomly selected person does not have asthma or a severe allergy? How did you determine this?

Scaffolding:

For students who are struggling with this concept, consider displaying a visual of the numerical representation of each problem (e.g., $300 - 50$, $19\,378\,102 - 4\,324\,929$, $100\% - 20\%$) and asking them, “What do all of these expressions have in common?”

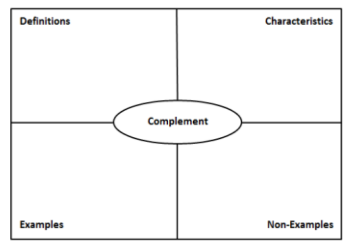
Discuss with students how these examples, or similar examples, involve finding the probability of the *complement of a given event.* Work with students in deriving the probabilities by thinking of the probability of people who are not in the given probability and connecting that probability to the fact that the two probabilities (or the probability of the given event and the probability of the complement) add up to 1.00 or 100%. Although students have previously worked with this idea, this lesson formalizes their understanding of complement and the probability of complementary events.

MP.8 Use this as an opportunity to have students look for a general method or rule for determining the probability of the complement of an event. Encourage students to try to write a rule, and ask them to share it with their neighbor.

- Based on the three examples we looked at, what is a general rule we could write to determine the probability of the complement of an event?

Scaffolding:

- Clarify that the term *complement* is not the same as *compliment*. Point out the difference in spelling between the terms, as well as their meanings.
- The *complement* of any event is the event that does not occur.
- A *compliment* is an expression of respect or admiration.
- Consider using visual displays and repeated choral readings to reinforce the mathematical meaning of the word.
- A Frayer model may also be used.



Example 1 (3 minutes): The Complement Rule

At this point, students have informally used the complement rule, but in this example it is given as a formula. The example serves as a quick illustration of the rule. Read through the example as a class, and then give students a moment to calculate the probability presented at the end of the example.

Example 1: The Complement Rule

In previous lessons, you have seen that to calculate the probability that an event *does not happen,* you can subtract the probability of the event from 1. If the event is denoted by A, then this rule can be written:

$$P(\text{not } A) = 1 - P(A).$$

For example, suppose that the probability that a particular flight is on time is 0.78. What is the probability that the flight is not on time?

$$P(\text{not on time}) = 1 - 0.78 = 0.22$$

Scaffolding:

- In addition to using Venn diagrams, if students have trouble identifying the intersection of two events, point out real-world examples like intersecting streets to connect students to the concept.
- Students may confuse the term *conditional* with *conditioner* or *condition*. Point out the difference in spelling and meaning.
- Again, the use of visuals and repeated choral readings helps reinforce these words.

Example 2 (6 minutes): Formula for Conditional Probability

The purpose of this example is to introduce the formula for conditional probability. A conditional probability is first calculated using a hypothetical 1000 table (as in previous lessons), and then the formula is shown to produce the same result.

When working through part (c) with the class, it would be helpful to illustrate the division with the aid of a Venn diagram so that students get a visual idea of what is being divided by what. (The probability of the intersection is being divided by the probability of B.)

Consider asking students to try solving the entire problem or parts and then discussing the results as a class.

Additionally, point out that $\frac{0.38}{0.43}$ is exactly the same thing as $\frac{380}{430}$ (the numerator and the denominator have been divided by $1{,}000$). By seeing this, students should see why the conditional probability formula is valid. (This is the main point of this example.)

Example 2: Formula for Conditional Probability

When a room is randomly selected in a downtown hotel, the probability that the room has a king-sized bed is 0.62, the probability that the room has a view of the town square is 0.43, and the probability that it has a king-sized bed *and* a view of the town square is 0.38. Let A be the event that the room has a king-sized bed, and let B be the event that the room has a view of the town square.

a. **What is the meaning of $P(A \text{ given } B)$ in this context?**

$P(A \text{ given } B)$ is the probability that a room known to have a view of the town square also has a king-sized bed.

b. **Use a hypothetical 1000 table to calculate $P(A \text{ given } B)$.**

	A (room has a king-sized bed)	Not A (room does not have a king-sized bed)	Total
B (room has a view of the town square)	380	50	430
Not B (room does not have a view of the town square)	240	330	570
Total	620	380	$1{,}000$

$$P(A \text{ given } B) = \frac{380}{430} \approx 0.884$$

c. **There is also a formula for calculating a conditional probability. The formula for conditional probability is**

$$P(A \text{ given } B) = \frac{P(A \text{ and } B)}{P(B)}.$$

Use this formula to calculate $P(A \text{ given } B)$, where the events A and B are as defined in this example.

$$P(A \text{ given } B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{0.38}{0.43} \approx 0.884$$

d. **How does the probability you calculated using the formula compare to the probability you calculated using the hypothetical 1000 table?**

The probabilities are the same.

Exercise 1 (15 minutes)

This exercise provides practice using the conditional probability formula. Additionally, in part (e), students are asked to compare a conditional and an unconditional probability and to provide an interpretation. In part (f), students are asked to recall the definition of independence in terms of equality of the conditional and unconditional probabilities.

Let students work with a partner and then confirm answers as a class, spending about 10 minutes total on the exercise. After confirming the answer to part (f), present the multiplication rule for independent events (approximately 5 minutes). Use this as an opportunity to informally assess student understanding of the lesson's outcomes.

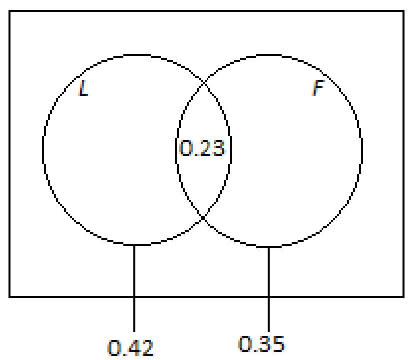
Exercise 1

A credit card company states that 42% of its customers are classified as long-term cardholders, 35% pay their bills in full each month, and 23% are long-term cardholders who also pay their bills in full each month. Let the event that a randomly selected customer is a long-term cardholder be L and the event that a randomly selected customer pays his bill in full each month be F.

a. **What are the values of $P(L)$, $P(F)$, and $P(L \text{ and } F)$?**

$P(L) = 0.42$, $P(F) = 0.35$, $P(L \text{ and } F) = 0.23$

b. **Draw a Venn diagram, and label it with the probabilities from part (a).**



c. **Use the conditional probability formula to calculate $P(L \text{ given } F)$. (Round your answer to the nearest thousandth.)**

$$P(L \text{ given } F) = \frac{P(L \text{ and } F)}{P(F)} = \frac{0.23}{0.35} \approx 0.657$$

d. **Use the conditional probability formula to calculate $P(F \text{ given } L)$. (Round your answer to the nearest thousandth.)**

$$P(F \text{ given } L) = \frac{P(F \text{ and } L)}{P(L)} = \frac{0.23}{0.42} \approx 0.548$$

e. **Which is greater, $P(F \text{ given } L)$ or $P(F)$? Explain why this is relevant.**

$P(F \text{ given } L) \approx 0.548$, and $P(F) = 0.35$; therefore, $P(F \text{ given } L)$ is larger than $P(F)$. This tells us that long-term cardholders are more likely to pay their bills in full each month than customers in general.

f. **Remember that two events A and B are said to be independent if $P(A \text{ given } B) = P(A)$. Are the events F and L independent? Explain.**

Note that there are several ways to answer this question. Here are three possibilities:

No, because $P(F \text{ given } L) \neq P(F)$.

No, because $P(L \text{ given } F) \neq P(L)$.

No, because $(L \text{ and } F) \neq P(L)P(F)$.

After confirming answers to part (f), introduce the multiplication rule for independent events.

- Events A and B are independent if and only if $P(A \text{ and } B) = P(A)P(B)$.

This means that if we know the events A and B are independent, then we can conclude that $P(A \text{ and } B) = P(A)P(B)$, and if we know that $P(A \text{ and } B) = P(A)P(B)$, then we can also conclude that the events A and B are independent. Consider asking students to state the meaning of this rule in their own words. The easiest way to explain this is as follows:

Scaffolding:

For students working above grade level, consider asking the following:
"Explain why this statement makes sense using conditional probabilities."

(The table below offers a sample explanation.)

If the events A and B are independent, then we know:	$P(A \text{ given } B) = P(A)$
Use the formula for conditional probability to replace $P(A \text{ given } B)$: $$P(A \text{ given } B) = \frac{P(A \text{ and } B)}{P(B)}$$	$$\frac{P(A \text{ and } B)}{P(B)} = P(A)$$
Now, isolate $P(A \text{ and } B)$ and conclude that:	$P(A \text{ and } B) = P(A)P(B)$

Example 3 (5 minutes): Using the Multiplication Rule for Independent Events

This is an example of the use of the multiplication rule for independent events. There are two ways of telling whether events are independent. Either it is obvious from the description of the problem (as in part (a)), or the question tells students that the events are independent (as in part (b)). Consider asking students to attempt to solve independently or with a neighbor, informally assessing understanding and offering guidance as necessary. When tackling part (a):

- Explain to students that the result for the number cube cannot possibly affect the result for the coin, and so the two events are independent.
- Explain that the result of any one roll of the number cube cannot have an effect on the results of the other two rolls.

Example 3: Using the Multiplication Rule for Independent Events

A number cube has faces numbered 1 through 6, and a coin has two sides, heads and tails.

The number cube will be rolled, and the coin will be flipped. Find the probability that the cube shows a 4 and the coin lands on heads. Because the events are independent, we can use the multiplication rules we just learned.

$$\left(\frac{1}{6}\right)\left(\frac{1}{2}\right) = \frac{1}{12}$$

If you toss the coin five times, what is the probability you will see a head on all five tosses?

$(0.5)(0.5)(0.5)(0.5)(0.5) = 0.03125$

If you tossed the coin five times and got five heads, would you think that this coin is a fair coin? Why or why not?

Although getting five heads is possible (about 3% of the time you would expect this), it is not likely; therefore, you would suspect that the coin is not fair.

If you roll the number cube three times, what is the probability that it will show 4 on all three throws?

$\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)\left(\frac{1}{6}\right) = \frac{1}{216} \approx 0.005$

If you rolled the number cube three times and got a 4 on all three rolls, would you think that this number cube is fair? Why or why not?

The probability of getting a 4 on all three rolls is very small. As a result, you suspect the number cube is not fair.

Suppose that the credit card company introduced in Exercise 1 states that when a customer is selected at random, the probability that the customer pays his bill in full each month is 0.35, the probability that the customer makes regular online purchases is 0.83, and these two events are independent. What is the probability that a randomly selected customer pays his bill in full each month *and* makes regular online purchases?

$(0.35)(0.83) = 0.2905$

Exercise 2 (5 minutes)

MP.4

This exercise provides practice with use of the multiplication rule for independent events. Students should think about *why* the events given are independent (in this case, because it is obvious from the problem description). Particularly in part (c), students have an opportunity to interpret their results in the context of the question and to reflect on whether the results make sense. Consider asking students to work independently on this exercise.

Exercise 2

A spinner has a pointer, and when the pointer is spun, the probability that it stops in the red section of the spinner is 0.25.

a. **If the pointer is spun twice, what is the probability that it will stop in the red section on both occasions?**

   $(0.25)(0.25) = 0.0625$

b. **If the pointer is spun four times, what is the probability that it will stop in the red section on all four occasions? (Round your answer to the nearest thousandth.)**

   $(0.25)(0.25)(0.25)(0.25) \approx 0.004$

c. **If the pointer is spun five times, what is the probability that it never stops on red? (Round your answer to the nearest thousandth.)**

   $(0.75)(0.75)(0.75)(0.75)(0.75) \approx 0.237$

Closing (2 minutes)

Ask students to summarize the main ideas of the lesson in writing or with a neighbor. Use this as an opportunity to informally assess comprehension of the lesson. The Lesson Summary below offers some important ideas that should be included.

- Does it make sense to consider the two events in Example 3, part (b), to be independent? (Events are that customer "pays bill in full each month" and "makes regular online purchases.")
  - *Yes, it seems feasible that long-term cardholders might be as likely to make regular online purchases as customers in general.*

Lesson Summary

For any event A, $P(\text{not } A) = 1 - P(A)$.

For any two events A and B, $P(A \text{ given } B) = \frac{P(A \text{ and } B)}{P(A)}$.

Events A and B are independent if and only if $P(A \text{ and } B) = P(A)P(B)$.

Exit Ticket (5 minutes)

Name ____ Date ____________________

Lesson 6: Probability Rules

Exit Ticket

1. Of the light bulbs available at a store, 42% are fluorescent, 23% are labeled as long life, and 12% are fluorescent *and* long life.
   a. A light bulb will be selected at random from the light bulbs at this store. Rounding your answer to the nearest thousandth where necessary, find the probability that
      i. The selected light bulb is not fluorescent.

      ii. The selected light bulb is fluorescent given that it is labeled as long life.

   b. Are the events "fluorescent" and "long life" independent? Explain.

2. When a person is selected at random from a very large population, the probability that the selected person is right-handed is 0.82. If three people are selected at random, what is the probability that
   a. They are all right-handed?

   b. None of them is right-handed?

Exit Ticket Sample Solutions

1. Of the light bulbs available at a store, 42% are fluorescent, 23% are labeled as long life, and 12% are fluorescent *and* long life.
   a. A light bulb will be selected at random from the light bulbs at this store. Rounding your answer to the nearest thousandth where necessary, find the probability that
      i. The selected light bulb is not fluorescent.

      $1 - 0.42 = 0.58$

      ii. The selected light bulb is fluorescent given that it is labeled as long life.

      $$P(\text{fluorescent given long life}) = \frac{P(\text{fluorescent and long life})}{P(\text{long life})} = \frac{0.12}{0.23} \approx 0.522$$

   b. Are the events "fluorescent" and "long life" independent? Explain.

   No. $P(\text{fluorescent given long life}) \neq P(\text{fluorescent})$

2. When a person is selected at random from a very large population, the probability that the selected person is right-handed is 0.82. If three people are selected at random, what is the probability that
   a. They are all right-handed?

   $(0.82)(0.82)(0.82) \approx 0.551$

   b. None of them is right-handed?

   $(0.18)(0.18)(0.18) \approx 0.006$

Problem Set Sample Solutions

1. When an avocado is selected at random from those delivered to a food store, the probability that it is ripe is 0.12, the probability that it is bruised is 0.054, and the probability that it is ripe and bruised is 0.019.
   a. Rounding your answers to the nearest thousandth where necessary, find the probability that an avocado randomly selected from those delivered to the store is
      i. Not bruised.

      $1 - 0.054 = 0.946$

      ii. Ripe given that it is bruised.

      $$P(\text{ripe given bruised}) = \frac{P(\text{ripe and bruised})}{P(\text{bruised})} = \frac{0.019}{0.054} \approx 0.352$$

      iii. Bruised given that it is ripe.

      $$P(\text{bruised given ripe}) = \frac{P(\text{bruised and ripe})}{P(\text{ripe})} = \frac{0.019}{0.12} \approx 0.158$$

   b. Which is larger, the probability that a randomly selected avocado is bruised given that it is ripe or the probability that a randomly selected avocado is bruised? Explain in words what this tells you.

   $P(\text{bruised given ripe}) = 0.158$ *and* $P(\text{bruised}) = 0.054$*. Therefore* $P(\text{bruised given ripe})$ *is greater than* $P(\text{bruised})$*, which tells you that ripe avocados are more likely to be bruised than avocados in general.*

c. Are the events "ripe" and "bruised" independent? Explain.

No, because $P(\text{bruised given ripe})$ is different from $P(\text{bruised})$.

2. Return to the probability information given in Problem 1. Complete the hypothetical 1000 table given below, and use it to find the probability that a randomly selected avocado is bruised given that it is not ripe. (Round your answer to the nearest thousandth.)

	Ripe	Not Ripe	Total
Bruised	19	35	54
Not Bruised	101	845	946
Total	120	880	$1,000$

$P(\text{bruised given not ripe}) = \frac{35}{880} \approx 0.040$

3. According to the U.S. census website (www.census.gov), based on the U.S. population in 2010, the probability that a randomly selected man is 65 or older is 0.114, and the probability that a randomly selected woman is 65 or older is 0.146. In the questions that follow, round your answers to the nearest thousandth:

a. If a man is selected at random and a woman is selected at random, what is the probability that both people selected are 65 or older? (Hint: Use the multiplication rule for independent events.)

$(0.114)(0.146) \approx 0.017$

b. If two men are selected at random, what is the probability that both of them are 65 or older?

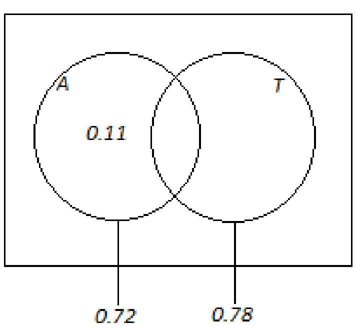
$(0.114)(0.114) \approx 0.013$

c. If two women are selected at random, what is the probability that neither of them is 65 or older?

If one woman is selected at random, the probability that she is not 65 or older is $1 - 0.146 = 0.854$. So, if two women are selected at random, the probability that neither of them is 65 or older is $(0.854)(0.854) \approx 0.729$.

4. In a large community, 72% of the people are adults, 78% of the people have traveled outside the state, and 11% are adults who have not traveled outside the state.

a. Using a Venn diagram or a hypothetical 1000 table, calculate the probability that a randomly selected person from the community is an adult and has traveled outside the state.



$P(\text{adult and traveled out of state}) = 0.72 - 0.11 = 0.61$

b. **Use the multiplication rule for independent events to decide whether the events "is an adult" and "has traveled outside the state" are independent.**

$P(\text{adult and traveled out of state}) = 0.61$

$P(\text{adult})P(\text{traveled out of state}) = (0.72)(0.78) = 0.5616$

Since these two quantities are not equal, the two events are not independent.

5. **In a particular calendar year, 10% of the registered voters in a small city are called for jury duty. In this city, people are selected for jury duty at random from all registered voters in the city, and the same individual cannot be called more than once during the calendar year.**

a. **What is the probability that a registered voter is not called for jury duty during a particular year?**

0.90

b. **What is the probability that a registered voter is called for jury duty two years in a row?**

$(0.10)(0.10) = 0.01$

6. **A survey of registered voters in a city in New York was carried out to assess support for a new school tax. 51% of the respondents supported the school tax. Of those with school-age children, 56% supported the school tax, while only 45% of those who did not have school-age children supported the school tax.**

a. **If a person who responded to this survey is selected at random, what is the probability that**

i. **The person selected supports the school tax?**

0.51

ii. **The person supports the school tax given that she does not have school-age children?**

0.45

b. **Are the two events "has school-age children" and "supports the school tax" independent? Explain how you know this.**

These two events are not independent because the probability of support given no school-age children is not the same as the probability of support.

c. **Suppose that 35% of those responding to the survey were over the age of 65 and that 10% of those responding to the survey were both over age 65 and supported the school tax. What is the probability that a randomly selected person who responded to this survey supported the school tax given that she was over age 65?**

$$\begin{aligned} P(\text{support given over age 65}) &= \frac{P(\text{support and over age 65})}{P(\text{over age 65})} \\ &= \frac{0.10}{0.35} \\ &\approx 0.286 \end{aligned}$$