Lesson 7: Probability Rules

Student Outcomes

- Students use the addition rule to calculate the probability of a union of two events.
- Students interpret probabilities in context.

Lesson Notes

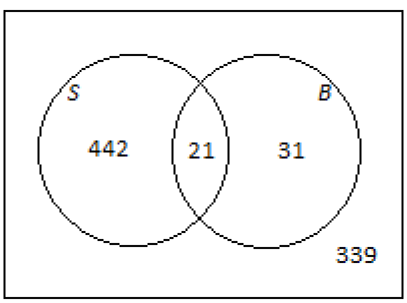
This lesson builds off of the probability rules presented in Lesson 6 and introduces the addition rule for calculating the probability of the union of two events. The general form of the rule is considered, as well as the special cases for disjoint and independent events. The use of Venn diagrams is encouraged throughout the lesson to illustrate problems.

Classwork

Opening (3 minutes)

Revisit the high school considered in the opening discussion of Lesson 5. Encourage students to work independently in finding the answer to the central question, "What is the number of students in the band or in organized sports?"

- 442 students participate in organized sports but do not play in the band,
- 31 students play in the band but do not participate in organized sports,
- 21 students participate in organized sports *and* play in the band, and
- 339 students neither participate in organized sports nor play in the band.



Use the Venn diagram to highlight the pieces involved in answering the following questions, as the answers to each of these questions are used to determine the number of students in the band *or* in organized sports:

How would you find the number of students in sports?

How would you find the number of students in band?

Point out to students that to answer each of the above questions, they had to add the 21 students involved in sports *and* band (the intersection) to find the total number of students in sports or to find the total number of students in band.

Discuss how the number of students in band or sports would be calculated if the summary of the school was presented differently. In particular, discuss with students how they would determine the number of students in band or sports if the description of the school was the following:

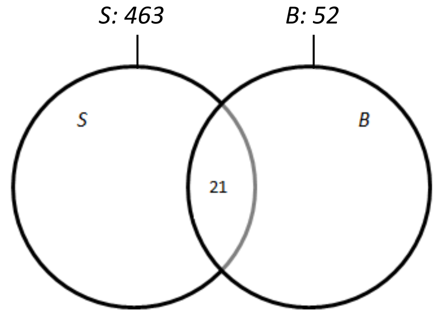
463 students are in sports,

52 students are in the band, and

21 students are in both sports and band.

The number of students in sports or the number of students in band (called the *union*) is $463 + 52 - 21$. The piece representing students in band and sports (or the intersection) is part of the total number of students in band, and it is also part of the total number of students in sports. As a result, if the number of students in band (52) and the number of students in sports (463) are added together, the 21 students in both band and sports are counted twice. As indicated, it is necessary to subtract the 21 students in both band and sports to make sure that these students are counted only once. Generalizing this as a probability of the union of two overlapping events is the focus of this lesson.

> *Scaffolding:*
>
> Ask students working above grade level to draw a Venn diagram to illustrate this scenario and determine how many students are in sports or in the band.
>
> For students working below grade level, use the Venn diagram below to illustrate this scenario.

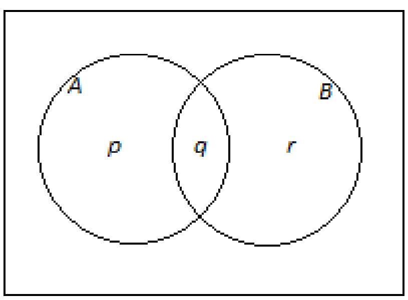


Exercise 1 (9 minutes)

Introduce the following addition rule to students. (This rule was informally illustrated with the above example and in several questions in the earlier lessons with two-way frequency tables.) The addition rule states that for any two events A and B,

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

To illustrate, draw a Venn diagram, denoting the probabilities of events as shown.



(Note that $p = P(A \text{ and not } B)$, $q = P(A \text{ and } B)$, and $r = P(B \text{ and not } A)$, but the labeling of the Venn diagram should be sufficient to communicate this.)

Therefore, $P(A) + P(B) - P(A \text{ and } B) = (p + q) + (q + r) - q$

$$= p + q + r$$

$$= P(A \text{ or } B).$$

> *Scaffolding:*
>
> For students who may be struggling with this concept, consider displaying and discussing several concrete examples in conjunction with the abstract representation.
>
> For example, using the case of a coin flip, with A represents the event of heads and B represents the event of tails:
>
> $$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$
> $$1 = 0.5 + 0.5 - 0$$
>
> Discuss the meaning of each component of the equation in context (e.g., 1 makes sense in this situation because it is certain that the coin either lands on heads or tails; 0 makes sense because it could not land on both heads and tails in one flip).
>
> For students working above grade level, consider encouraging them to determine the addition rule independently by asking, "What is a general rule for determining $P(A \text{ or } B)$? Use the example from the Opening to determine your answer."

Indicate to students that $P(A \text{ or } B)$ is $p+q+r$ using the Venn diagram directly. Note that when the probability of the events A or B were added together, the probability of q was added twice; therefore, the addition rule indicates that q (the intersection) is subtracted from the sum of the two events to make sure it is not added twice.

Exercise 1 is a straightforward application of the addition rule. Since Exercises 1 and 2 are students' first experience using the addition rule, consider having students work in pairs. Use this as an opportunity to informally assess student understanding of the addition rule.

Exercise 1

When a car is brought to a repair shop for a service, the probability that it will need the transmission fluid replaced is 0.38, the probability that it will need the brake pads replaced is 0.28, and the probability that it will need both the transmission fluid and the brake pads replaced is 0.16. Let the event that a car needs the transmission fluid replaced be T and the event that a car needs the brake pads replaced be B.

a. **What are the values of the following probabilities?**

i. $P(T)$ 0.38

ii. $P(B)$ 0.28

iii. $P(T \text{ and } B)$ 0.16

b. **Use the addition rule to find the probability that a randomly selected car needs the transmission fluid or the brake pads replaced.**

$P(T \text{ or } B) = P(T) + P(B) - P(T \text{ and } B) = 0.38 + 0.28 - 0.16 = 0.5$

Exercise 2 (5 minutes)

Here students are asked to use the addition rule in conjunction with the multiplication rule for independent events.

Exercise 2

Josie will soon be taking exams in math and Spanish. She estimates that the probability she passes the math exam is 0.9, and the probability that she passes the Spanish exam is 0.8. She is also willing to assume that the results of the two exams are independent of each other.

a. **Using Josie's assumption of independence, calculate the probability that she passes both exams.**

$P(\text{passes both}) = (0.9)(0.8) = 0.72$

b. **Find the probability that Josie passes at least one of the exams. (Hint: Passing at least one of the exams is passing math *or* passing Spanish.)**

$$P(\text{passes math or Spanish}) = P(\text{passes math}) + P(\text{passes Spanish}) - P(\text{passes both})$$
$$= 0.9 + 0.8 - 0.72 = 0.98$$

Example 1 (7 minutes): Use of the Addition Rule for Disjoint Events

Introduce the idea of *disjoint events* and how the addition rule for disjoint events is different from the addition rule for events that had an intersection. While discussing the following examples with students, ask them how these events differ from the previous examples:

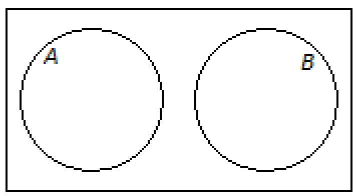
- An animal hospital has 5 dogs and 3 cats out of 10 animals in the hospital. What is the probability that an animal selected at random is a dog or a cat?

- At a certain high school, 100 students are involved in an after-school community service program. Students can only sign up for one project. Currently, 25 students are involved in cleaning up nearby parks, 20 students are tutoring elementary students in mathematics, and the rest of the students are working at helping out at a community recreational center. What is the probability that a randomly selected student is involved in cleaning up nearby parks or tutoring elementary students in mathematics?

The above examples are different in that they do not have any students in the intersection. Students would indicate that the addition rule of two events would not have a piece that needs to be subtracted. The probability of randomly selecting a dog or a cat is $\frac{5}{10}+\frac{3}{10}$. The probability of randomly selecting a student involved in cleaning a nearby park or tutoring elementary students in mathematics is $\frac{25}{100}+\frac{20}{100}$.

Summarize the following with students:

- Two events are said to be *disjoint* if they have no outcomes in common. So, if the events A and B are disjoint, the Venn diagram looks like this.



Another way of describing disjoint events is by saying that they cannot both happen at the same time. Continue discussing with students other examples.

If a number cube has faces numbered 1–6, and the number cube is rolled once, then the events "the result is even" and "the result is a 5" are disjoint (since *even* and 5 cannot both happen on a single roll), but the events "the result is even" and "the result is greater than 4" are not (since getting a 6 results in both events occurring).

It would be a good idea at this stage to provide some other examples of disjoint and non-disjoint events so that students get used to the meaning of the term.

> *Scaffolding:*
>
> Point out to students that the meaning of *disjoint* can be found by examining the prefix *dis-* and the word *joint*. Remind students that *dis-* means *not*. The stem *joint* has several other meanings that might need to be explained or explored.

If A and B are disjoint, then $P(A \text{ and } B)=0$. So, the addition rule for disjoint events can be written as

$$P(A \text{ or } B)=P(A)+P(B).$$

Now, work through the example presented in the lesson as a class. This is a straightforward application of the addition rule for disjoint events.

Example 1: Use of the Addition Rule for Disjoint Events

A set of 40 cards consists of the following:

- **10 black cards showing squares**
- **10 black cards showing circles**
- **10 red cards showing X's**
- **10 red cards showing diamonds**

A card will be selected at random from the set. Find the probability that the card is black or shows a diamond.

The events "is black" and "shows a diamond" are disjoint since there are no black cards that show diamonds. So,

$$P(\text{black or diamond}) = P(\text{black}) + P(\text{diamond}) = \frac{20}{40} + \frac{10}{40} = \frac{30}{40} = \frac{3}{4}.$$

Example 2 (4 minutes): Combining Use of the Multiplication and Addition Rules

The addition rule for disjoint events is often used in conjunction with the multiplication rule for independent events. This example illustrates this.

When tackling part (b), point out to students that the three events—red shows 6 and blue shows 5, red shows 5 and blue shows 6, red shows 6 and blue shows 6—are disjoint. This is why the probabilities are added together.

Example 2: Combining Use of the Multiplication and Addition Rules

A red cube has faces labeled 1 through 6, and a blue cube has faces labeled in the same way. The two cubes are rolled. Find the probability of each event.

a. **Both cubes show 6's.**

$$P(\text{red shows 6 and blue shows 6}) = P(\text{red shows 6})P(\text{blue shows 6}) = \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36}$$

b. **The total score is at least 11.**

$$P(\text{total is at least 11}) = P(\text{red shows 6 and blue shows 5}) + P(\text{red shows 5 and blue shows 6}) + P(\text{red shows 6 and blue shows 6})$$

$$P(\text{total is at least 11}) = \frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6} + \frac{1}{6}\cdot\frac{1}{6} = \frac{1}{36} + \frac{1}{36} + \frac{1}{36} = \frac{1}{12}$$

Exercise 3 (7 minutes)

This exercise provides additional practice with the ideas introduced in Example 2. Part (c) is a little more complex than any part of Example 2, and part (d) requires students to recall the complement rule. Have students first work the solutions independently. Then, have students compare and discuss solutions with a partner. Ask students to describe and compare with each other how they found their solutions. This is an opportunity for students to show persistence in solving a problem.

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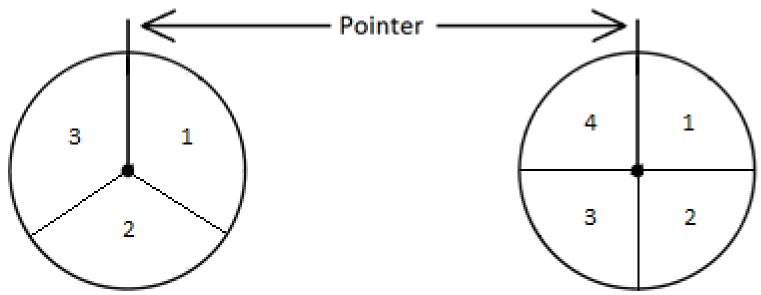
Exercise 3



The diagram above shows two spinners. For the first spinner, the scores 1, 2, and 3 are equally likely, and for the second spinner, the scores 1, 2, 3, and 4 are equally likely. Both pointers will be spun. Writing your answers as fractions in lowest terms, find the probability of each event.

a. **The total of the scores on the two spinners is 2.**

$$P(\text{total} = 2) = P(1,1) = \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$$

b. **The total of the scores on the two spinners is 3.**

$$P(\text{total} = 3) = P(1,2) + P(2,1) = \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12} + \frac{1}{12} = \frac{2}{12} = \frac{1}{6}$$

c. **The total of the scores on the two spinners is 5.**

$$\begin{aligned} P(\text{total} = 5) &= P(1,4) + P(2,3) + P(3,2) \\ &= \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} + \frac{1}{3} \cdot \frac{1}{4} \\ &= \frac{1}{12} + \frac{1}{12} + \frac{1}{12} \\ &= \frac{3}{12} \\ &= \frac{1}{4} \end{aligned}$$

d. **The total of the scores on the two spinners is not 5.**

$$P(\text{total is not } 5) = 1 - P(\text{total is } 5) = 1 - \frac{1}{4} = \frac{3}{4}$$

Closing (3 minutes)

Review the difference between independent and disjoint events.

- Define independent and disjoint events.
  - *Two events are independent if knowing that one event has occurred does not change the probability that the other event has occurred. Two events are said to be disjoint if they have no outcomes in common.*

Ask students to summarize the main ideas of the lesson with a neighbor or in writing. Use this as an opportunity to informally assess comprehension of the lesson. The Lesson Summary below offers some important ideas that should be included.

Lesson Summary

The addition rule states that for any two events A and B, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$.

The addition rule can be used in conjunction with the multiplication rule for independent events: Events A and B are independent if and only if $P(A \text{ and } B) = P(A)P(B)$.

Two events are said to be *disjoint* if they have no outcomes in common. If A and B are disjoint events, then $P(A \text{ or } B) = P(A) + P(B)$.

The addition rule for disjoint events can be used in conjunction with the multiplication rule for independent events.

Exit Ticket (7 minutes)

Name ____ Date____________________

Lesson 7: Probability Rules

Exit Ticket

1. When a call is received at an airline's call center, the probability that it comes from abroad is 0.32, and the probability that it is to make a change to an existing reservation is 0.38.
   a. Suppose that you are told that the probability that a call is both from abroad and is to make a change to an existing reservation is 0.15. Calculate the probability that a randomly selected call is either from abroad or is to make a change to an existing reservation.

   b. Suppose now that you are *not* given the information in part (a), but you are told that the events "the call is from abroad" and "the call is to make a change to an existing reservation" are independent. What is the probability that a randomly selected call is either from abroad or is to make a change to an existing reservation?

2. A golfer will play two holes of a course. Suppose that on each hole the player will score 3, 4, 5, 6, or 7, with these five scores being equally likely. Find the probability, and explain how the answer was determined that the player's total score for the two holes will be
   a. 14.

   b. 12.

Exit Ticket Sample Solutions

1. When a call is received at an airline's call center, the probability that it comes from abroad is 0.32, and the probability that it is to make a change to an existing reservation is 0.38.

   a. Suppose that you are told that the probability that a call is both from abroad and is to make a change to an existing reservation is 0.15. Calculate the probability that a randomly selected call is either from abroad or is to make a change to an existing reservation.

   $$\begin{aligned} P(\text{abroad or change}) &= P(\text{abroad}) + P(\text{change}) - P(\text{abroad and change}) \\ &= 0.32 + 0.38 - 0.15 \\ &= 0.55 \end{aligned}$$

   b. Suppose now that you are *not* given the information in part (a), but you are told that the events "the call is from abroad" and "the call is to make a change to an existing reservation" are independent. What is the probability that a randomly selected call is either from abroad or is to make a change to an existing reservation?

   $$\begin{aligned} P(\text{abroad and change}) &= P(\text{abroad})P(\text{change}) \\ &= (0.32)(0.38) \\ &= 0.1216 \end{aligned}$$

   So,

   $$\begin{aligned} P(\text{abroad or change}) &= P(\text{abroad}) + P(\text{change}) - P(\text{abroad and change}) \\ &= 0.32 + 0.38 - 0.1216 \\ &= 0.5784. \end{aligned}$$

2. A golfer will play two holes of a course. Suppose that on each hole the player will score 3, 4, 5, 6, or 7, with these five scores being equally likely. Find the probability, and explain how the answer was determined that the player's total score for the two holes will be

   a. 14.

   The only way to have a total score of 14 is if the player scores 7 on each hole, which is $P(7) \cdot P(7)$. Each of the scores is equally likely, so $P(7) = \frac{1}{5} = 0.2$.

   $$P(\text{total} = 14) = P(7) \cdot P(7) = (0.2)(0.2) = 0.04$$

   b. 12.

   There are three ways to have a total score of 12. The player could score 7 on the first hole and 5 on the second. The player could score 6 on the first hole and 6 on the second. Finally, the player could score 5 on the first hole and 7 on the second. Again, all of the scores are equally likely, so $P(5) = P(6) = P(7) = \frac{1}{5} = 0.2$.

   $$\begin{aligned} P(\text{total} = 12) &= P(7,5) + P(6,6) + P(5,7) \\ &= (0.2)(0.2) + (0.2)(0.2) + (0.2)(0.2) \\ &= 0.04 + 0.04 + 0.04 \\ &= 0.12 \end{aligned}$$

Problem Set Sample Solutions

1. Of the works of art at a large gallery, 59% are paintings, and 83% are for sale. When a work of art is selected at random, let the event that it is a painting be A and the event that it is for sale be B.

   a. What are the values of $P(A)$ and $P(B)$?

      $P(A) = 0.59$

      $P(B) = 0.83$

   b. Suppose you are told that $P(A \text{ and } B) = 0.51$. Find $P(A \text{ or } B)$.

      $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = 0.59 + 0.83 - 0.51 = 0.91$

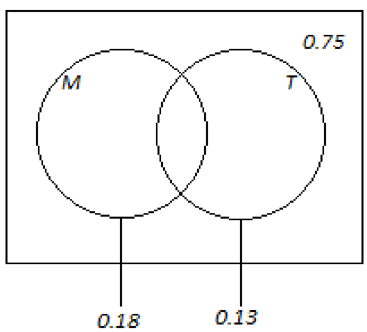
   c. Suppose now that you are not given the information in part (b), but you are told that the events A and B are independent. Find $P(A \text{ or } B)$.

      $$P(A \text{ and } B) = P(A)P(B) = (0.59)(0.83) = 0.4897$$

      So, $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) = 0.59 + 0.83 - 0.4897 = 0.9303.$

2. A traveler estimates that, for an upcoming trip, the probability of catching malaria is 0.18, the probability of catching typhoid is 0.13, and the probability of catching neither of the two diseases is 0.75.

   a. Draw a Venn diagram to represent this information.



   b. Calculate the probability of catching both of the diseases.

      $$P(M \text{ or } T) = 1 - 0.75 = 0.25$$

      By the addition rule:

      $$P(M \text{ or } T) = P(M) + P(T) - P(M \text{ and } T)$$
      $$0.25 = 0.18 + 0.13 - P(M \text{ and } T)$$
      $$0.25 = 0.31 - P(M \text{ and } T)$$
      $$P(M \text{ and } T) = 0.06$$

   c. Are the events "catches malaria" and "catches typhoid" independent? Explain your answer.

      $P(M \text{ and } T) = 0.06$

      $P(M)P(T) = (0.18)(0.13) = 0.0234$

      Since these quantities are different, the two events are not independent.

3. A deck of 40 cards consists of the following:
   - 10 black cards showing squares, numbered 1–10
   - 10 black cards showing circles, numbered 1–10
   - 10 red cards showing X's, numbered 1–10
   - 10 red cards showing diamonds, numbered 1–10

   A card will be selected at random from the deck.

   a. i. Are the events "the card shows a square" and "the card is red" disjoint? Explain.

   Yes. There is no red card that shows a square.

   ii. Calculate the probability that the card will show a square or will be red.

   $$P(\text{square or red}) = P(\text{square}) + P(\text{red}) = \frac{10}{40} + \frac{20}{40} = \frac{30}{40} = \frac{3}{4}$$

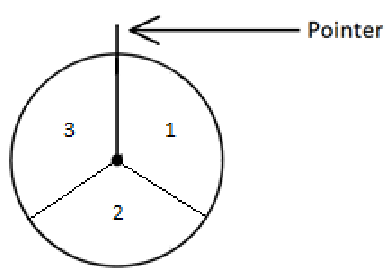
   b. i. Are the events "the card shows a 5" and "the card is red" disjoint? Explain.

   No. There are red fives in the deck.

   ii. Calculate the probability that the card will show a 5 or will be red.

   $$P(5\text{ or red}) = P(5) + P(\text{red}) - P(5\text{ and red}) = \frac{4}{40} + \frac{20}{40} - \frac{2}{40} = \frac{22}{40} = \frac{11}{20}$$

4. The diagram below shows a spinner. When the pointer is spun, it is equally likely to stop on 1, 2, or 3. The pointer will be spun three times. Expressing your answers as fractions in lowest terms, find the probability, and explain how the answer was determined that the total of the values from all thee spins is



   a. 9.

   The only way to get a total of 9 is to spin a 3, 3 times. Since the probability of spinning a 3 is $\frac{1}{3}$,

   $$P(\text{total is } 9) = P(3,3,3) = \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \frac{1}{27}.$$

   b. 8.

   There are 3 ways to get a total of 8. Since the probability of spinning a 1, 2, and 3 are all equally likely $\left(\frac{1}{3}\right)$:

   $$P(\text{total is } 8) = P(3,3,2) + P(3,2,3) + P(2,3,3)$$
   $$= \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}$$
   $$= \frac{1}{27} + \frac{1}{27} + \frac{1}{27}$$
   $$= \frac{3}{27}$$
   $$= \frac{1}{9}$$

c. 7.

There are 6 ways to get a total of 7. Since the probability of spinning a 1, 2, and 3 are all equally likely $\left(\frac{1}{3}\right)$:

$$\begin{aligned} P(\text{total is } 7) &= P(3,3,1) + P(3,1,3) + P(1,3,3) + P(3,2,2) + P(2,3,2) + P(2,2,3) \\ &= 6\left(\frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3}\right) \\ &= \frac{6}{27} \\ &= \frac{2}{9} \end{aligned}$$

5. A number cube has faces numbered 1 through 6, and a coin has two sides—heads and tails. The number cube will be rolled once, and the coin will be flipped once. Find the probabilities of the following events. (Express your answers as fractions in lowest terms.)

a. The number cube shows a 6.

$\frac{1}{6}$

b. The coin shows heads.

$\frac{1}{2}$

c. The number cube shows a 6, and the coin shows heads.

$\frac{1}{6}\cdot\frac{1}{2} = \frac{1}{12}$

d. The number cube shows a 6, or the coin shows heads.

$$\begin{aligned} P(6 \text{ or heads}) &= P(6) + P(\text{heads}) - P(6 \text{ and heads}) \\ &= \frac{1}{6} + \frac{1}{2} - \frac{1}{12} = \frac{2}{12} + \frac{6}{12} - \frac{1}{12} = \frac{7}{12} \end{aligned}$$

6. Kevin will soon be taking exams in math, physics, and French. He estimates the probabilities of his passing these exams to be as follows:

- Math: 0.9
- Physics: 0.8
- French: 0.7

Kevin is willing to assume that the results of the three exams are independent of each other. Find the probability of each event.

a. Kevin will pass all three exams.

$(0.9)(0.8)(0.7) = 0.504$

b. Kevin will pass math but fail the other two exams.

$(0.9)(0.2)(0.3) = 0.054$

c. **Kevin will pass exactly one of the three exams.**

$$\begin{aligned} P(\text{passes exactly one}) &= P(\text{passes math, fails physics, fails French}) \\ &\quad + P(\text{fails math, passes physics, fails French}) \\ &\quad + P(\text{fails math, fails physics, passes French}) \\ &= (0.9)(0.2)(0.3) + (0.1)(0.8)(0.3) + (0.1)(0.2)(0.7) \\ &= 0.092 \end{aligned}$$